OPERATOR SPACE EMBEDDING OF L_q INTO L_p

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Introduction

The idea of replacing functions by linear operators, the process of quantization, goes back to the foundations of quantum mechanics and has a great impact in mathematics. This applies for instance to representation theory, operator algebra, noncommutative geometry, quantum and free probability or operator space theory. The quantization of measure theory leads to the theory of L_p spaces defined over general von Neumann algebras, so called *noncommutative L_p spaces*. This theory was initiated by Segal, Dixmier and Kunze in the fifties and continued years later by Haagerup, Fack, Kosaki and many others. We refer to the recent survey [39] for a complete exposition. In this paper we will investigate noncommutative L_p spaces in the language of noncommutative Banach spaces, so called *operator spaces*. The theory of operator spaces took off in 1988 with Ruan's work [44]. Since then, it has been developed by Blecher/Paulsen, Effros/Ruan and Pisier as a noncommutative generalization of Banach space theory, see e.g. [4, 29, 34]. In his book [33] on vector valued noncommutative L_p spaces, Pisier considered a distinguished operator space structure on L_p. In fact, the right category when dealing with noncommutative L_p is in many aspects that of operator spaces. Indeed, this has become clear in the last years by recent results on noncommutative martingales and related topics. In this paper, we prove a fundamental structure theorem of L_p spaces in the category of operator spaces, solving a problem formulated by Gilles Pisier.

Main result. *Let $1 \le p < q \le 2$ and let $\mathcal{M}$ be a von Neumann algebra. Then, there exists a sufficiently large von Neumann algebra $\mathcal{A}$ and a completely isomorphic embedding of $L_q(\mathcal{M})$ into $L_p(\mathcal{A})$, where both spaces are equipped with their respective natural operator space structures. Moreover, we have*

- **(a)** *If $\mathcal{M}$ is* QWEP*, we can choose $\mathcal{A}$ to be* QWEP.
- **(b)** *If $\mathcal{M}$ is hyperfinite, we can choose $\mathcal{A}$ to be hyperfinite.*

In order to put our result in the right context, let us stress the interaction between harmonic analysis, probability and Banach space theory carried out mostly in the 70's. Based on previous results by Beck, Grothendieck, Lévy, Orlicz, etc... probabilistic methods in Banach spaces became the heart of the work developed by Kwapień, Maurey, Pisier, Rosenthal and many others. A fundamental motivation for this new field relied on the embedding theory of classical L_p spaces. This theory was born in 1966 with the seminal paper [2] of Bretagnolle, Dacunha-Castelle and Krivine. They constructed an isometric embedding of L_q into L_p for $1 \le p < q \le 2$, a Banach space version of our main result. The simplest form of such embedding

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was known to Lévy and is given by

$$\Big(\sum_{k=1}^\infty |\alpha_k|^q \Big)^{\frac1q} = \Big\| \sum_{k=1}^\infty \alpha_k \, \theta_k \Big\|_{L_1(\Omega)}, \tag{1}$$

for scalars $(\alpha_k)_{k \ge 1}$ and where $(\theta_k)_{k \ge 1}$ is a suitable sequence of independent q-stable random variables in $L_1(\Omega)$ for some probability space (Ω, μ). In other words, we have the relation

$$\mathbb{E} \exp \big(i \sum\nolimits_k \alpha_k \theta_k \big) = \exp \big(- c_q \sum\nolimits_k |\alpha_k|^q \big).$$

More recently, it has been discovered a parallel connection between operator space theory and quantum probability. The operator space version of Grothendieck theorem by Pisier and Shlyakhtenko [37] and the embedding of OH [11] require tools from free probability. In this context we should replace the θ_k's by suitable operators so that (1) holds with matrix-valued coefficients $\alpha_1, \alpha_2, \ldots$. To that aim, we develop new tools in quantum probability and construct an operator space version of q-stable random variables. To formulate this quantized form of (1) we need some basic results of Pisier's theory [33]. The most natural operator space structure on ℓ_∞ comes from the diagonal embedding $\ell_\infty \hookrightarrow \mathcal{B}(\ell_2)$. The natural structure on ℓ_1 is given by operator space duality, while the spaces ℓ_p are defined by means of the complex interpolation method [32] for operator spaces. Let us denote by $(\delta_k)_{k \ge 1}$ the unit vector basis of ℓ_q. If $\widehat{\otimes}$ denotes the operator space projective tensor product and S_p stands for the Schatten p-class over ℓ_2, it was shown in [33] that

$$\Big\| \sum_{k=1}^\infty a_k \otimes \delta_k \Big\|_{S_1 \widehat{\otimes} \ell_q} = \inf_{a_k = \alpha b_k \beta} \|\alpha\|_{S_{2q'}} \Big(\sum_{k=1}^\infty \|b_k\|_{S_q}^q \Big)^{\frac1q} \|\beta\|_{S_{2q'}}.$$

The answer to Pisier's problem for ℓ_q reads as follows.

Theorem A. *If $1 < q \le 2$, there exists a sufficiently large von Neumann algebra $\mathcal{A}$ and a sequence $(x_k)_{k \ge 1}$ in $L_1(\mathcal{A})$ such that the equivalence below holds for any family $(a_k)_{k \ge 1}$ of trace class operators*

$$\inf_{a_k = \alpha b_k \beta} \|\alpha\|_{2q'} \Big(\sum_{k=1}^\infty \|b_k\|_q^q \Big)^{\frac1q} \|\beta\|_{2q'} \sim_c \Big\| \sum_{k=1}^\infty a_k \otimes x_k \Big\|_{L_1(\mathcal{A} \bar{\otimes} \mathcal{B}(\ell_2))}.$$

This gives a completely isomorphic embedding of ℓ_q into $L_1(\mathcal{A})$. Moreover, the sequence $x_1, x_2, \ldots$ provides an operator space version of a q-stable process and motivates a cb-embedding theory of L_p spaces. A particular case of Theorem A is the recent construction [11] of a cb-embedding of Pisier's operator Hilbert space OH into a von Neumann algebra predual. In other words, a complete embedding of ℓ_2 (with its natural operator space structure) into a noncommutative L_1 space, see also Pisier's paper [36] for a shorter proof and Xu's alternative construction [52]. Other related results appear in [13, 35, 37, 51], while semi-complete embeddings between (vector-valued) L_p spaces can be found in [15, 26]. All these papers will play a role in our analysis.

Let us sketch the simplest construction (a more elaborated one is needed to prove the stability of hyperfiniteness) which leads to this operator space version of q-stable random variables. A key ingredient in our proof is the notion of the Haagerup tensor product $\otimes_h$. We first note that ℓ_q is the diagonal subspace of the

Schatten class S_q. According to [33], S_q can be written as the Haagerup tensor product of its first column and first row subspaces $S_q = C_q \otimes_h R_q$. Moreover, using a simple generalization of "Pisier's exercise" (see Exercise 7.9 in Pisier's book [34]) we have

$$\begin{array}{rcl} C_q & \hookrightarrow_{cb} & \big(R \oplus \mathrm{OH}\big)/graph(\Lambda_1)^\perp, \\ R_q & \hookrightarrow_{cb} & \big(C \oplus \mathrm{OH}\big)/graph(\Lambda_2)^\perp, \end{array} \tag{2}$$

with $\Lambda_1 : C \to \mathrm{OH}$ and $\Lambda_2 : R \to \mathrm{OH}$ suitable injective, closed, densely-defined operators with dense range and where $\hookrightarrow_{cb}$ denotes a cb-embedding, see [11, 36, 51] for related results. By [11] and duality, it suffices to see that $graph(\Lambda_1) \otimes_h graph(\Lambda_2)$ is cb-isomorphic to a cb-complemented subspace of $\mathcal{A}(\mathrm{OH})$. By the injectivity of the Haagerup tensor product, we note that $graph(\Lambda_1) \otimes_h graph(\Lambda_2)$ is an intersection of four spaces. Let us explain this in detail. By discretization we may assume that $\Lambda_j = \mathsf{d}_{\lambda^j} = \sum_k \lambda_k^j e_{kk}$ is a diagonal operator on ℓ_2. In fact, by a simple complementation argument, it is no restriction to assume that the eigenvalues are the same for $j = 1, 2$. Thus we may write the Haagerup tensor product above as follows

$$\mathcal{J}_{\infty,2} = graph(\mathsf{d}_\lambda) \otimes_h graph(\mathsf{d}_\lambda) = \Big(C \cap \ell_2^{oh}(\lambda)\Big) \otimes_h \Big(R \cap \ell_2^{oh}(\lambda)\Big),$$

where $\ell_2^{oh}(\lambda)$ is a weighted form of OH according to the action of d_λ. That is

$$\begin{array}{rcl} C \cap \ell_2^{oh}(\lambda) & = & \Big\{(e_{i1}, \lambda_i e_{i1}) \, \big| \, i \geq 1\Big\} \subset C \oplus \mathrm{OH}, \\ R \cap \ell_2^{oh}(\lambda) & = & \Big\{(e_{1j}, \lambda_j e_{1j}) \, \big| \, j \geq 1\Big\} \subset R \oplus \mathrm{OH}. \end{array}$$

The symbol ∞ in $\mathcal{J}_{\infty,2}$ is used because we shall consider L_p versions of these spaces along the paper. The number 2 denotes that this space arises as a 'middle point' in the sense of interpolation theory between two related $\mathcal{J}$-spaces, see Section 3 below for more details. Now, regarding $\mathsf{d}_\lambda^4 = \sum_k \lambda_k^4 e_{kk}$ as the density d_ψ of some normal strictly semifinite faithful weight ψ on $\mathcal{B}(\ell_2)$, the space $\mathcal{J}_{\infty,2}$ splits up into a 4-term intersection space. In other words, we find

$$\mathcal{J}_{\infty,2}(\psi) = \big(C \otimes_h R\big) \cap \big(C \otimes_h \mathrm{OH}\big) d_\psi^{\frac14} \cap d_\psi^{\frac14} \big(\mathrm{OH} \otimes_h R\big) \cap d_\psi^{\frac14} \big(\mathrm{OH} \otimes_h \mathrm{OH}\big) d_\psi^{\frac14}.$$

The norm of x in $\mathcal{J}_{\infty,2}(\psi)$ is given by

$$\max \Big\{ \|x\|_{B(\ell_2)}, \big\|x d_\psi^{\frac14}\big\|_{C \otimes_h \mathrm{OH}}, \big\|d_\psi^{\frac14} x\big\|_{\mathrm{OH} \otimes_h R}, \big\|d_\psi^{\frac14} x d_\psi^{\frac14}\big\|_{\mathrm{OH} \otimes_h \mathrm{OH}} \Big\}.$$

The two middle terms are not as unusual as it might seem

$$\begin{array}{rcl} \big\|d_\psi^{\frac14}(x_{ij})\big\|_{\mathrm{M}_m(\mathrm{OH} \otimes_h R)} & = & \displaystyle\sup_{\|\alpha\|_{S_4^m} \leq 1} \Big\| d_\psi^{\frac14} \Big(\sum_{k=1}^m \alpha_{ik} x_{kj} \Big) \Big\|_{L_4(\mathrm{M}_m \otimes \mathcal{B}(\ell_2))}, \\ \big\|(x_{ij}) d_\psi^{\frac14}\big\|_{\mathrm{M}_m(C \otimes_h \mathrm{OH})} & = & \displaystyle\sup_{\|\beta\|_{S_4^m} \leq 1} \Big\| \Big(\sum_{k=1}^m x_{ik} \beta_{kj} \Big) d_\psi^{\frac14} \Big\|_{L_4(\mathrm{M}_m \otimes \mathcal{B}(\ell_2))}. \end{array} \tag{3}$$

Let us now assume that we just try to embed the finite-dimensional space S_q^m. By approximation, it suffices to consider only finitely many eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and according to the results from [13], we can take $n \sim m \log m$. In this case we rename ψ by ψ_n and we may easily assume that $\mathrm{tr}(d_{\psi_n}) = \sum_k \lambda_k^4 = \mathrm{k}_n$ is an integer.

Therefore, we may consider the following state on $\mathcal{B}(\ell_2^n)$

$$\varphi_n(x) = \frac{1}{\mathrm{k}_n} \sum_{k=1}^n \lambda_k^4 \, x_{kk}.$$

In this particular case, the space $\mathcal{J}_{\infty,2}(\psi_n)$ can be obtained using free probability. More precisely, we may identify it as a subspace of $L_\infty(\mathcal{A}; \mathrm{OH}_{\mathrm{k}_n})$, which in turn is the space obtained by complex interpolation $L_\infty(\mathcal{A}; \mathrm{OH}_{\mathrm{k}_n}) = [C_{\mathrm{k}_n}(\mathcal{A}), R_{\mathrm{k}_n}(\mathcal{A})]_{1/2}$ where $R_{\mathrm{k}_n}(\mathcal{A})$ and $C_{\mathrm{k}_n}(\mathcal{A})$ denote the row and column subspaces of $\mathrm{M}_{\mathrm{k}_n}(\mathcal{A})$.

Theorem B. *Let $\mathcal{A} = \mathsf{A}_1 * \mathsf{A}_2 * \cdots * \mathsf{A}_{\mathrm{k}_n}$ be the reduced free product of k_n copies of $\mathcal{B}(\ell_2^n) \oplus \mathcal{B}(\ell_2^n)$ equipped with the state $\frac{1}{2}(\varphi_n \oplus \varphi_n)$. If $\pi_k : \mathsf{A}_j \to \mathcal{A}$ denotes the canonical embedding into the j-th component of $\mathcal{A}$, the mapping*

$$u_n : x \in \mathcal{J}_{\infty,2}(\psi_n) \mapsto \sum_{j=1}^{\mathrm{k}_n} \pi_j(x, -x) \otimes \delta_j \in L_\infty(\mathcal{A}; \mathrm{OH}_{\mathrm{k}_n})$$

is a cb-embedding with cb-complemented image and constants independent of n.

Theorem B and its generalization for arbitrary von Neumann algebras is a very recent result from [16]. However, the proof given there is rather long and quite technical, see Section 3 for further details. In order to be more self-contained, we provide a second proof only using elementary tools from free probability. We think this alternative approach is of independent interest. Now, by duality we obtain a cb-embedding of $\ell_q^m \hookrightarrow_{cb} S_q^m$ into $L_1(\mathcal{A}; \mathrm{OH}_{\mathrm{k}_n})$. Then we use the cb-embedding [11] of OH into a von Neumann algebra predual. Using an ultraproduct procedure, the desired cb-embedding of ℓ_q into $L_1(\mathcal{A})$ is obtained. In fact, what we shall prove is a far reaching generalization of Theorem A. Namely, the same result holds replacing C_q and R_q by subspaces of quotients of $R \oplus \mathrm{OH}$ and $C \oplus \mathrm{OH}$ respectively.

Theorem C. *Let X_1 be a subspace of a quotient $R \oplus \mathrm{OH}$ and let X_2 be a subspace of a quotient of $C \oplus \mathrm{OH}$. Then, there exist some* QWEP *algebra $\mathcal{A}$ and a cb-embedding*

$$\mathrm{X}_1 \otimes_h \mathrm{X}_2 \hookrightarrow_{cb} L_1(\mathcal{A}).$$

Now we may explain how the paper is organized. In Section 1 we just prove the complete embeddings (2). This is a simple consequence of Pisier's exercise 7.9 in [34] and Xu's generalization [51]. However, we have decided to include the proof since it serves as a model in our construction of the cb-embedding for general von Neumann algebras. In Section 2 we concentrate on the simplest form of our main result by proving Theorems A, B, C. This is done in part to motivate a new class of noncommutative function spaces introduced in Section 3. We will state (in terms of these new spaces) the main result of [16], a further generalization of Theorem B and a key tool in the proof of our main result. This is a free analogue of a generalization of Rosenthal's inequality, as we shall explain in Section 3. The whole Section 4 is entirely devoted to the proof of our main result. We first construct a cb-embedding of S_q into $L_p(\mathcal{A})$ for some von Neumann algebra $\mathcal{A}$. This is quite similar to our argument sketched above and provides an L_p generalization of Theorem C, but this construction does not preserve hyperfiniteness. The argument to fix this is quite involved and requires recent techniques from [13] and [14]. We first apply a transference argument, via a noncommutative Rosenthal type inequality in L_1 for identically distributed random variables, to replace freeness in our construction by some sort of noncommutative independence. This allows to avoid free product von

Neumann algebras and use tensor products instead. Then we combine the algebraic central limit theorem with the notion of noncommutative Poisson random measure to eliminate the use of ultraproducts in the process. After these modifications in our original argument, it is easily seen that hyperfiniteness is preserved. This more involved construction of the cb-embedding is the right one to analyze the finite dimensional case. In other words, we estimate the dimension of $\mathcal{A}$ in terms of the dimension of $\mathcal{M}$, see Remark 4.21 below for details. The proof for general von Neumann algebras requires to consider a 'generalized' Haagerup tensor product since we are not in the discrete case anymore.

We conclude with some comments related to our main result. In the category of Banach spaces, noncommutative versions of a p-stable process were studied in [9] and further analyzed in [14] to construct Banach space embeddings between noncommutative L_p spaces. If $0 < p < q < 2$ and we write $\mathcal{R}$ for the hyperfinite II_1 factor, the main embedding result there asserts that the space $L_q(\mathcal{R}\bar{\otimes}\mathcal{B}(\ell_2))$ embeds isometrically into $L_p(\mathcal{R})$. One of the principal techniques in the proof is a noncommutative version of the notion of Poisson random measure, which will also be instrumental in this paper. On the other hand, the cb-embedding theory of L_p spaces presents significant differences. Indeed, in sharp contrast with the classical theory, it was proved in [8] that the operator space OH does not embed completely isomorphically into any L_p space for $2 < p < \infty$. Moreover, after [35] we know that there is no possible cb-embedding of OH into the predual of a semifinite von Neumann algebra. As it is to be expected, this also happens in our main result and justifies the relevance of type III von Neumann algebras in the subject.

Theorem D. *If $1 \le p < q \le 2$ and ℓ_q cb-embeds into $L_p(\mathcal{A})$, $\mathcal{A}$ is not semifinite.*

Unfortunately, the proof of this result is out of the scope of this paper and will be the subject of a forthcoming publication. In fact, the proof for the case $p = 1$ is much harder than the case $1 < p < q$ and requires the use of a noncommutative version of Rosenthal theorem [43], recently obtained in [17]. Our results there are closely related to this paper and complement Pisier's paper [30] and some recent results of Randrianantoanina [40]. After a quick look at the main results in [2, 9], the problem of constructing an *isometric* cb-embedding of L_q into L_p arises in a natural way. This remains an open problem.

Background and notation. We shall assume that the reader is familiar with those branches of operator algebra related to the theories of operator spaces and noncommutative L_p spaces. The recent monographs [4] and [34] on operator spaces contain more than enough information for our purposes. We shall work over general von Neumann algebras so that we use Haagerup's definition [6] of L_p, see also Terp's excellent exposition of the subject [47]. As is well known, Haagerup L_p spaces have trivial intersection and thereby do not form an interpolation scale. However, the complex interpolation method will be a basic tool in this paper. This is solved using Kosaki's definition [23] of L_p. We also refer to Chapter 1 in [16] or to the survey [39] for a quick review of Haagerup's and Kosaki's definitions of noncommutative L_p spaces and the compatibility between them. In particular, using such compatibility, we shall use in what follows the complex interpolation method without further details. The basics on von Neumann algebras and Tomita's modular theory required to work with these notions can be found in Kadison/Ringrose books [22]. There

are some other topics related to noncommutative L_p spaces that will be frequently used in the paper. The main properties of normal faithful conditional expectations over Haagerup L_p spaces can be found in [19] and [46]. We shall also assume certain familiarity with Pisier's theory of vector-valued non-commutative L_p spaces [33] and Voiculescu's free probability theory [49]. Along the text we shall find some other topics such as certain noncommutative function spaces, some recent inequalities for free random variables, a noncommutative version of a Poisson process, etc... In all these cases our exposition intends to be self-contained.

We shall follow the standard notation in the subject. Anyway, let us say a few words on our terminology. The symbols (δ_k) and (e_{ij}) will denote the unit vector basis of ℓ_2 and $\mathcal{B}(\ell_2)$ respectively. The letters $\mathcal{A}, \mathcal{M}$ and $\mathcal{N}$ are reserved to denote von Neumann algebras. Almost all the time, the inclusions $\mathcal{N} \subset \mathcal{M} \subset \mathcal{A}$ will hold. We shall use φ and ϕ to denote normal faithful (*n.f.* in short) states, while the letter ψ will be reserved for normal strictly semifinite faithful (*n.s.s.f.* in short) weights. Inner products and duality brackets will be anti-linear on the first component and linear on the second component. As usual, given an operator space X we shall write $\mathrm{M}_m(\mathrm{X})$ for the space of $m \times m$ matrices with entries in X and we shall equip it with the norm of the minimal tensor product $\mathrm{M}_m \otimes_{\min} \mathrm{X}$. Similarly, the X-valued Schatten p-class over M_m will be denoted by $S_p^m(\mathrm{X})$. Accordingly, $L_p(\mathcal{M};\mathrm{X})$ will stand for the X-valued L_p space over $\mathcal{M}$. Given $\gamma > 0$, we shall write $\gamma\mathrm{X}$ to denote the space X equipped with the norm $\|x\|_{\gamma\mathrm{X}} = \gamma\|x\|_{\mathrm{X}}$. In particular, if $\mathcal{M}$ is a finite von Neumann algebra equipped with a finite weight ψ, we shall usually write $\psi = \mathrm{k}\varphi$ with $\mathrm{k} = \psi(1)$ so that φ becomes a state on $\mathcal{M}$. In this situation, the associated L_p space will be denoted as $\mathrm{k}^{1/p}L_p(\mathcal{M})$, so that the L_p norm is calculated using the state φ. Any new or non-standard terminology will be properly introduced in the text.

1. On "Pisier's exercise"

We begin by proving a generalization of Exercise 7.9 in [34]. This result became popular after Pisier applied it in [36] to obtain a simpler way to cb-embed OH into the predual of a von Neumann algebra. In fact, our argument is quite close to the one given in [51] for a similar result and might be known to experts. Nevertheless we include it here for completeness, since it will be used below and mainly because it will also serve as a model for our construction in the non-discrete case. Let us set some notation. Given a Hilbert space $\mathcal{H}$, we shall write $\mathcal{H}_r = \mathcal{B}(\mathcal{H}, \mathbb{C})$ and $\mathcal{H}_c = \mathcal{B}(\mathbb{C}, \mathcal{H})$ for the row and column quantizations on $\mathcal{H}$. Moreover, given $1 \le p \le \infty$ we shall use the following terminology

$$\mathcal{H}_{r_p} = \big[\mathcal{H}_r, \mathcal{H}_c\big]_{\frac{1}{p}} \quad \text{and} \quad \mathcal{H}_{c_p} = \big[\mathcal{H}_c, \mathcal{H}_r\big]_{\frac{1}{p}}.$$

There are two particular cases for which we use another terminology. When $\mathcal{H} = \ell_2$, we shall use (R, C, R_p, C_p) instead. Moreover, when the Hilbert space is $L_2(\mathcal{M})$ for some von Neumann algebra $\mathcal{M}$, we shall write $L_2^{r_p}(\mathcal{M})$ and $L_2^{c_p}(\mathcal{M})$. In the same fashion, $\mathcal{H}_{oh}$ and $L_2^{oh}(\mathcal{M})$ stand for the operator Hilbert space structures. Given two operator spaces X_1 and X_2, the expression $\mathrm{X}_1 \simeq_{cb} \mathrm{X}_2$ means that there exists a complete isomorphism between them. We shall write $\mathrm{X}_1 \in \mathcal{QS}(\mathrm{X}_2)$ to denote that X_1 is completely isomorphic to a quotient of a subspace of X_2. Let $\mathcal{S}$ denote the

strip

$$\mathcal{S} = \Big\{ z \in \mathbb{C} \, \big| \, 0 < \mathrm{Re}(z) < 1 \Big\}$$

and let $\partial \mathcal{S} = \partial_0 \cup \partial_1$ be the partition of its boundary into

$$\partial_0 = \Big\{ z \in \mathbb{C} \, \big| \, \mathrm{Re}(z) = 0 \Big\} \quad \text{and} \quad \partial_1 = \Big\{ z \in \mathbb{C} \, \big| \, \mathrm{Re}(z) = 1 \Big\}.$$

Given $0 < \theta < 1$, let μ_θ be the harmonic measure of the point $z = \theta$. This is a probability measure on $\partial \mathcal{S}$ (with density given by the Poisson kernel in the strip) that can be written as $\mu_\theta = (1-\theta)\mu_0 + \theta \mu_1$, with μ_j being probability measures supported by ∂_j and such that

$$f(\theta) = \int_{\partial \mathcal{S}} f \, d\mu_\theta \tag{1.1}$$

for any bounded analytic function $f : \mathcal{S} \to \mathbb{C}$ extended non-tangentially to $\partial \mathcal{S}$. Now, before proving the announced result, we need to set a formula describing the norm of certain kind of vector-valued noncommutative L_p space. A more detailed account of these expressions will be given at the beginning of Section 3. Given $2 \le p \le \infty$ and $0 < \theta < 1$, the norm of $x = \sum_k x_k \otimes \delta_k$ in $\big[S_p(C_p), S_p(R_p) \big]_\theta$ is given by

$$\sup \Big\{ \Big(\sum\nolimits_k \big\| \alpha x_k \beta \big\|_{S_2}^2 \Big)^{\frac12} \, \big| \, \|\alpha\|_{S_u}, \|\beta\|_{S_v} \le 1 \Big\} \tag{1.2}$$

where the indices (u,v) are determined by

$$(1/u, 1/v) = (\theta/q, (1-\theta)/q) \quad \text{with} \quad 1/2 = 1/p + 1/q.$$

Of course, this formula trivially generalizes for the norm in the space

$$\big[S_p(\mathcal{H}_{c_p}), S_p(\mathcal{H}_{r_p}) \big]_\theta.$$

Lemma 1.1. *If $1 \le p < q \le 2$, we have*

$$R_q \in \mathcal{QS}\big(R_p \oplus_2 \mathrm{OH}\big) \quad \textit{and} \quad C_q \in \mathcal{QS}\big(C_p \oplus_2 \mathrm{OH}\big).$$

Proof. We only prove the first assertion since the arguments for both are the same. In what follows we fix $0 < \theta < 1$ determined by the relation $R_q = [R_p, \mathrm{OH}]_\theta$. In other words, we have $1/q = (1-\theta)/p + \theta/2$. According to the complex interpolation method and its operator space extension [32], given a compatible couple $(\mathrm{X}_0, \mathrm{X}_1)$ of operator spaces we define $\mathcal{F}(\mathrm{X}_0, \mathrm{X}_1)$ as the space of bounded analytic functions $f : \mathcal{S} \to \mathrm{X}_0 + \mathrm{X}_1$ and we equip it with the following norm

$$\|f\|_{\mathcal{F}(\mathrm{X}_0,\mathrm{X}_1)} = \Big((1-\theta) \, \|f_{|\partial_0}\|_{L_2(\partial_0;\mathrm{X}_0)}^2 + \theta \, \|f_{|\partial_1}\|_{L_2(\partial_1;\mathrm{X}_1)}^2 \Big)^{\frac12}.$$

Then, the complex interpolation space $\mathrm{X}_\theta = [\mathrm{X}_0, \mathrm{X}_1]_\theta$ can be defined as the space of all $x \in \mathrm{X}_0 + \mathrm{X}_1$ such that there exists a function $f \in \mathcal{F}(\mathrm{X}_0, \mathrm{X}_1)$ with $f(\theta) = x$. We equip X_θ with the norm

$$\|x\|_{\mathrm{X}_\theta} = \inf \Big\{ \|f\|_{\mathcal{F}(\mathrm{X}_0,\mathrm{X}_1)} \, \big| \, f \in \mathcal{F}(\mathrm{X}_0, \mathrm{X}_1) \text{ and } f(\theta) = x \Big\}.$$

In our case we set $\mathrm{X}_0 = R_p$ and $\mathrm{X}_1 = \mathrm{OH}$. If we define

$$\mathcal{H} = (1-\theta)^{\frac12} L_2(\partial_0; \ell_2) \quad \text{and} \quad \mathcal{K} = \theta^{\frac12} L_2(\partial_1; \ell_2),$$

it turns out that $\mathcal{F}(R_p, \mathrm{OH})$ can be regarded (via Poisson integration) as a subspace of $\mathcal{H} \oplus_2 \mathcal{K}$. Moreover, we equip $\mathcal{F}(R_p, \mathrm{OH})$ with the operator space structure inherited from $\mathcal{H}_{r_p} \oplus_2 \mathcal{K}_{oh}$. Then, we define the mapping

$$\mathcal{Q} : f \in \mathcal{F}(R_p, \mathrm{OH}) \mapsto f(\theta) \in R_q.$$

The assertion follows from the fact that $\mathcal{Q}$ is a complete metric surjection. Indeed, in that case we have $R_q \simeq_{cb} \mathcal{F}(R_p, \mathrm{OH})/\ker \mathcal{Q}$, which is a quotient of a subspace of $R_p \oplus_2 \mathrm{OH}$. In order to see that $\mathcal{Q}$ is a complete surjection, it suffices to see that the map $id_{S_{p'}} \otimes \mathcal{Q} : S_{p'}(\mathcal{F}(R_p, \mathrm{OH})) \to S_{p'}(R_q)$ is a metric surjection. We begin by showing that $id_{S_{p'}} \otimes \mathcal{Q}$ is contractive. Let $f \in S_{p'}(\mathcal{F}(R_p, \mathrm{OH}))$ be of norm < 1 and let us write

$$f(\theta) = \sum\nolimits_k f_k(\theta) \otimes \delta_k \in S_{p'}(R_q).$$

To compute the norm of $f(\theta)$ we note that

$$S_{p'}(R_q) = \big[S_{p'}(C_{p'}), S_{p'}(R_{p'}) \big]_\eta \quad \text{with} \quad 1/q = (1-\eta)/p + \eta/p'.$$

Then it follows from (1.2) that

$$\|f(\theta)\|_{S_{p'}(R_q)} = \sup \Big\{ \Big(\sum\nolimits_k \big\| \alpha f_k(\theta) \beta \big\|_{S_2}^2 \Big)^{\frac12} \ \big| \ \|\alpha\|_{S_{2r/\eta}}, \|\beta\|_{S_{2r/(1-\eta)}} \le 1 \Big\} \tag{1.3}$$

where $1/2r = 1/2 - 1/p' = 1/p - 1/2$. Moreover, it is clear that we can restrict the supremum above to all α and β in the positive parts of their respective unit balls. Taking this restriction in consideration, we define

$$g(z) = \sum\nolimits_k g_k(z) \otimes \delta_k \quad \text{with} \quad g_k(z) = \alpha^{\frac{z}{\theta}} f_k(z) \beta^{\frac{2-z}{2-\theta}}.$$

The g_k's are analytic in $\mathcal{S}$ and take values in S_2. Thus, we have the identity

$$\begin{aligned} & \Big(\sum\nolimits_k \big\| \alpha f_k(\theta) \beta \big\|_{S_2}^2 \Big)^{\frac12} \\ = & \ \Big(\sum\nolimits_k \|g_k(\theta)\|_{S_2}^2 \Big)^{\frac12} \\ = & \ \Big((1-\theta) \int_{\partial_0} \sum\nolimits_k \|g_k(z)\|_{S_2}^2 d\mu_0 + \theta \int_{\partial_1} \sum\nolimits_k \|g_k(z)\|_{S_2}^2 d\mu_1 \Big)^{\frac12}. \end{aligned} \tag{1.4}$$

The contractivity of $id_{S_{p'}} \otimes \mathcal{Q}$ will follow from

$$\int_{\partial_0} \sum\nolimits_k \|g_k(z)\|_{S_2}^2 d\mu_0 \ \le \ \|f_{|\partial_0}\|_{S_{p'}(L_2^{r_p}(\partial_0; \ell_2))}^2, \tag{1.5}$$

$$\int_{\partial_1} \sum\nolimits_k \|g_k(z)\|_{S_2}^2 d\mu_1 \ \le \ \|f_{|\partial_1}\|_{S_{p'}(L_2^{oh}(\partial_1; \ell_2))}^2. \tag{1.6}$$

Indeed, if we combine (1.3) and (1.4) with the operator space structure defined on $\mathcal{F}(R_p, \mathrm{OH})$, it turns out that inequalities (1.5) and (1.6) are exactly what we need. To prove (1.5) we observe that $2\eta = \theta$ and $r' = p'/2$, so that

$$\begin{aligned} \int_{\partial_0} \sum\nolimits_k \|g_k(z)\|_{S_2}^2 d\mu_0 & = \int_{\partial_0} \sum\nolimits_k \|f_k(z) \beta^{\frac{2}{2-\theta}}\|_{S_2}^2 d\mu_0 \\ & = \int_{\partial_0} \sum\nolimits_k \mathrm{tr} \big(f_k(z)^* f_k(z) \beta^{\frac{1}{1-\eta}} (\beta^{\frac{1}{1-\eta}})^* \big) d\mu_0 \\ & \le \|\beta^{\frac{2}{1-\eta}}\|_{S_r} \Big\| \int_{\partial_0} \sum\nolimits_k f_k(z)^* f_k(z) d\mu_0 \Big\|_{S_{r'}}. \end{aligned}$$

This gives

$$\int_{\partial_0} \sum_k \|g_k(z)\|_{S_2}^2 d\mu_0 \le \|\beta\|_{S_{2r/(1-\eta)}}^{\frac{2}{1-\eta}} \Big\| \Big(\int_{\partial_0} \sum_k f_k(z)^* f_k(z) d\mu_0 \Big)^{\frac{1}{2}} \Big\|_{S_{p'}}^2 .$$

The first term on the right is ≤ 1. Hence, we have

$$\int_{\partial_0} \sum_k \|g_k(z)\|_{S_2}^2 d\mu_0 \quad \le \quad \|f_{|\partial_0}\|_{S_{p'}(L_2^{c_{p'}}(\partial_0;\ell_2))}^2 = \|f_{|\partial_0}\|_{S_{p'}(L_2^{r_p}(\partial_0;\ell_2))}^2 .$$

This proves (1.5) while for (1.6) we proceed in a similar way

$$\begin{aligned}
&\int_{\partial_1} \sum_k \|g_k(z)\|_{S_2}^2 d\mu_1 \\
&\quad = \int_{\partial_1} \sum_k \|\alpha^{\frac{1}{\theta}} f_k(z) \beta^{\frac{1}{2-\theta}}\|_{S_2}^2 d\mu_1 \\
&\quad = \int_{\partial_1} \sum_k \|\alpha^{\frac{1}{2\eta}} f_k(z) \beta^{\frac{1}{2-2\eta}}\|_{S_2}^2 d\mu_1 \\
&\quad \le \sup \Big\{ \int_{\partial_1} \sum_k \|a f_k(z) b\|_{S_2}^2 d\mu_1 \ \big| \ \|a\|_{S_{4r}}, \|b\|_{S_{4r}} \le 1 \Big\} \\
&\quad = \sup \Big\{ \Big\| a \Big(\sum_k f_{k|\partial_1} \otimes \delta_k \Big) b \Big\|_{S_2(L_2^{oh}(\partial_1;\ell_2))}^2 \ \big| \ \|a\|_{S_{4r}}, \|b\|_{S_{4r}} \le 1 \Big\}.
\end{aligned}$$

According to (1.2), this proves (1.6) and we have a contraction. Reciprocally, given $x \in S_{p'}(R_q)$ of norm < 1, we are now interested on finding $f \in S_{p'}(\mathcal{F}(R_p, \mathrm{OH}))$ such that $f(\theta) = x$ and $\|f\|_{S_{p'}(\mathcal{F}(R_p,\mathrm{OH}))} \le 1$. Since

$$[S_{p'}(R_p), S_{p'}(\mathrm{OH})]_\theta = S_{p'}(R_q)$$

there must exists $f \in \mathcal{F}(S_{p'}(R_p), S_{p'}(\mathrm{OH}))$ such that $f(\theta) = x$ and

$$\|f\|_{\mathcal{F}(S_{p'}(R_p),S_{p'}(\mathrm{OH}))} = \Big((1-\theta) \|f_{|\partial_0}\|_{L_2(\partial_0;S_{p'}(R_p))}^2 + \theta \|f_{|\partial_1}\|_{L_2(\partial_1;S_{p'}(\mathrm{OH}))}^2 \Big)^{\frac{1}{2}} \le 1.$$

Therefore, it remains to see that

$$\|f_{|\partial_0}\|_{S_{p'}(L_2^{r_p}(\partial_0;\ell_2))} \quad \le \quad \|f_{|\partial_0}\|_{L_2(\partial_0;S_{p'}(R_p))}, \tag{1.7}$$

$$\|f_{|\partial_1}\|_{S_{p'}(L_2^{oh}(\partial_1;\ell_2))} \quad \le \quad \|f_{|\partial_1}\|_{L_2(\partial_1;S_{p'}(\mathrm{OH}))}. \tag{1.8}$$

However, the identities below are clear by now

$$\begin{aligned}
\|f_{|\partial_0}\|_{S_{p'}(L_2^{r_p}(\partial_0;\ell_2))} &= \Big\| \Big(\int_{\partial_0} \sum_k f_k(z)^* f_k(z) d\mu_0 \Big)^{\frac{1}{2}} \Big\|_{S_{p'}}, \\
\|f_{|\partial_0}\|_{L_2(\partial_0;S_{p'}(R_p))} &= \Big(\int_{\partial_0} \Big\| \Big(\sum_k f_k(z)^* f_k(z) \Big)^{\frac{1}{2}} \Big\|_{S_{p'}}^2 d\mu_0 \Big)^{\frac{1}{2}}.
\end{aligned}$$

In particular, (1.7) follows automatically. On the other hand, (1.2) gives

$$\begin{aligned}
\|f_{|\partial_1}\|_{S_{p'}(L_2^{oh}(\partial_1;\ell_2))} &= \sup_{\|\alpha\|_{4r}, \|\beta\|_{4r} \le 1} \Big(\int_{\partial_1} \sum_k \|\alpha f_k(z) \beta\|_{S_2}^2 d\mu_1 \Big)^{\frac{1}{2}}, \\
\|f_{|\partial_1}\|_{L_2(\partial_1;S_{p'}(\mathrm{OH}))} &= \Big(\int_{\partial_1} \sup_{\|\alpha\|_{4r}, \|\beta\|_{4r} \le 1} \sum_k \|\alpha f_k(z) \beta\|_{S_2}^2 d\mu_1 \Big)^{\frac{1}{2}}.
\end{aligned}$$

Thus, inequality (1.8) also follows easily and $\mathcal{Q}$ is a complete metric surjection. □

Remark 1.2. If $1 \le p_0 \le p \le p_1 \le \infty$, it is also true that
$$R_p \in \mathcal{QS}\big(R_{p_0} \oplus_2 R_{p_1}\big) \quad \text{and} \quad C_p \in \mathcal{QS}\big(C_{p_0} \oplus_2 C_{p_1}\big).$$
The arguments to prove it are the same. However, we have preferred to state and prove the particular case with $p_1 = 2$ for the sake of clarity, since we think of Lemma 1.1 as a model to follow when dealing with non-discrete algebras.

Remark 1.3. In Lemma 1.1 we have obtained
$$R_q \simeq_{cb} \mathcal{F}(R_p, \mathrm{OH})/\ker \mathcal{Q} \in \mathcal{QS}(R_p \oplus_2 \mathrm{OH}),$$
$$C_q \simeq_{cb} \mathcal{F}(C_p, \mathrm{OH})/\ker \mathcal{Q} \in \mathcal{QS}(C_p \oplus_2 \mathrm{OH}).$$
However, it will be convenient in the sequel to observe that $\ker \mathcal{Q}$ can be regarded in both cases as the annihilator of the graph of certain linear operator, see (2). Recall that for a given a linear map between Hilbert spaces $\Lambda : \mathcal{K}_1 \to \mathcal{K}_2$ with domain $\mathsf{dom}(\Lambda)$, its graph is defined by
$$graph(\Lambda) = \Big\{ (x_1, x_2) \in \mathcal{K}_1 \oplus_2 \mathcal{K}_2 \, \big| \, x_1 \in \mathsf{dom}(\Lambda) \text{ and } x_2 = \Lambda(x_1) \Big\}.$$
Let us consider for instance the situation with R_q. We first observe that $\mathcal{F}(R_p, \mathrm{OH})$ is the graph of an injective, closed, densely-defined operator Λ with dense range. This operator is given
$$\Lambda(f_{|\partial_0}) = f_{|\partial_1}$$
for all $f \in \mathcal{F}(R_p, \mathrm{OH})$. Λ is well defined and injective by Poisson integration due to the analyticity of elements in $\mathcal{F}(R_p, \mathrm{OH})$. On the other hand, $\ker \mathcal{Q}$ is the subspace of $\mathcal{F}(R_p, \mathrm{OH})$ composed of functions f vanishing at $z = \theta$. Then, it easily follows from (1.1) that $\ker \mathcal{Q}$ is the annihilator of $\mathcal{F}(R_{p'}, \mathrm{OH}) = graph(\Lambda)$ regarded as a subspace of
$$(1-\theta)^{\frac12} L_2^{r_{p'}}(\partial_0; \ell_2) \oplus_2 \theta^{\frac12} L_2^{oh}(\partial_1; \ell_2).$$

2. The simplest case

Given $1 < q \le 2$, we construct a completely isomorphic embedding of S_q into the predual of a QWEP (not yet hyperfinite) von Neumann algebra. This is the simplest case of our main result and will serve as a motivation for the general case. We start with the proof of a generalized form of Theorem B for arbitrary von Neumann algebras, although we just use for the moment the discrete version as stated in the Introduction. The general formulation will be instrumental when dealing with non-discrete algebras. In the second part of this section, we prove Theorem C and deduce our cb-embedding via Lemma 1.1. Theorem A and the subsequent family of operator space q-stable random variables arise by injecting the space ℓ_q into the diagonal of the Schatten class S_q.

2.1. **Free harmonic analysis.** Our starting point is a von Neumann algebra $\mathcal{M}$ equipped with a *n.f.* state φ and associated density d_φ. Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$. According to Takesaki [46], the existence and uniqueness of a *n.f.* conditional expectation $\mathsf{E} : \mathcal{M} \to \mathcal{N}$ is equivalent to the invariance of $\mathcal{N}$ under the action of the modular automorphism group σ_t^φ associated to $(\mathcal{M}, \varphi)$. Moreover, in that case E is φ-invariant and following Connes [3] we have $\mathsf{E} \circ \sigma_t^\varphi = \sigma_t^\varphi \circ \mathsf{E}$. In what follows, we assume these properties in all subalgebras considered. Now we set $\mathsf{A}_k = \mathcal{M} \oplus \mathcal{M}$ and define $\mathcal{A}$ to be the reduced amalgamated free product $*_\mathcal{N} \mathsf{A}_k$ of the family $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ over the subalgebra $\mathcal{N}$. Note that our notation $*_\mathcal{N} \mathsf{A}_k$

for reduced amalgamated free products does not make explicit the dependence on the conditional expectations $\mathsf{E}_k : \mathsf{A}_k \to \mathcal{N}$, given by $\mathsf{E}_k(a,b) = \frac{1}{2}\mathsf{E}(a) + \frac{1}{2}\mathsf{E}(b)$. The following is the operator-valued version [11, 18] of Voiculescu inequality [48], for which we need to introduce the mean-zero subspaces

$$\mathring{\mathsf{A}}_k = \Big\{ x \in \mathsf{A}_k \, \Big| \, \mathsf{E}_k(a_k) = 0 \Big\}.$$

Lemma 2.1. *If $a_k \in \mathring{\mathsf{A}}_k$ for $1 \le k \le n$ and $\mathsf{E}_\mathcal{N} : \mathcal{A} \to \mathcal{N}$ stands for the conditional expectation of $\mathcal{A}$ onto $\mathcal{N}$, the following equivalence of norms holds with constants independent of n*

$$\Big\| \sum_{k=1}^n a_k \Big\|_\mathcal{A} \sim \max_{1 \le k \le n} \|x_k\|_{\mathsf{A}_k} + \Big\| \Big(\sum_{k=1}^n \mathsf{E}_\mathcal{N}(a_k a_k^*) \Big)^{\frac12} \Big\|_\mathcal{N} + \Big\| \Big(\sum_{k=1}^n \mathsf{E}_\mathcal{N}(a_k^* a_k) \Big)^{\frac12} \Big\|_\mathcal{N}.$$

Moreover, we also have

$$\Big\| \big(\sum_{k=1}^n a_k a_k^* \big)^{\frac12} \Big\|_\mathcal{A} \sim \max_{1 \le k \le n} \|a_k\|_{\mathsf{A}_k} + \Big\| \Big(\sum_{k=1}^n \mathsf{E}_\mathcal{N}(a_k a_k^*) \Big)^{\frac12} \Big\|_\mathcal{N},$$

$$\Big\| \big(\sum_{k=1}^n a_k^* a_k \big)^{\frac12} \Big\|_\mathcal{A} \sim \max_{1 \le k \le n} \|a_k\|_{\mathsf{A}_k} + \Big\| \Big(\sum_{k=1}^n \mathsf{E}_\mathcal{N}(a_k^* a_k) \Big)^{\frac12} \Big\|_\mathcal{N}.$$

Proof. For the first inequality we refer to [11]. The others can be proved in a similar way. Alternatively, both can be deduced from the first one. Indeed, using the identity

$$\Big\| \big(\sum_{k=1}^n a_k a_k^* \big)^{\frac12} \Big\|_\mathcal{A} = \Big\| \sum_{k=1}^n a_k \otimes e_{1k} \Big\|_{\mathrm{M}_n(\mathcal{A})}$$

and recalling the isometric isomorphism

$$\mathrm{M}_n\big(*_\mathcal{N} \mathsf{A}_k \big) = *_{\mathrm{M}_n(\mathcal{N})} \mathrm{M}_n(\mathsf{A}_k),$$

we may apply Voiculescu's inequality over the triple

$$\Big(\mathrm{M}_n(\mathcal{A}), \mathrm{M}_n(\mathsf{A}_k), \mathrm{M}_n(\mathcal{N}) \Big).$$

Taking $\widetilde{\mathsf{E}}_\mathcal{N} = id_{\mathrm{M}_n} \otimes \mathsf{E}_\mathcal{N}$, the last term disappears because

$$\Big\| \Big(\sum_{k=1}^n \widetilde{\mathsf{E}}_\mathcal{N}\big((a_k \otimes e_{1k})^* (a_k \otimes e_{1k}) \big) \Big)^{\frac12} \Big\|_{\mathrm{M}_n(\mathcal{N})} = \sup_{1 \le k \le n} \big\| \mathsf{E}_\mathcal{N}(a_k^* a_k)^{\frac12} \big\|_\mathcal{N} \le \sup_{1 \le k \le n} \|a_k\|_{\mathsf{A}_k}.$$

The third equivalence follows by taking adjoints. The proof is complete. □

Let $\pi_k : \mathsf{A}_k \to \mathcal{A}$ denote the embedding of A_k into the k-th component of $\mathcal{A}$. Given $x \in \mathcal{M}$, we shall write x_k as an abbreviation of $\pi_k(x, -x)$. Note that x_k is mean-zero. In the following we shall use with no further comment the identities $\mathsf{E}_\mathcal{N}(x_k x_k^*) = \mathsf{E}(xx^*)$ and $\mathsf{E}_\mathcal{N}(x_k^* x_k) = \mathsf{E}(x^*x)$. We will mostly work with identically distributed variables. In other words, given $x \in \mathcal{M}$ we shall work with the sequence $x_k = \pi_k(x, -x)$ for $1 \le k \le n$. In terms of the last equivalences in Lemma 2.1, we may consider the following norms

$$\begin{aligned} \|x\|_{\mathcal{R}_{\infty,1}^n} &= \max \Big\{ \|x\|_\mathcal{M}, \sqrt{n} \, \big\| \mathsf{E}(xx^*)^{\frac12} \big\|_\mathcal{N} \Big\}, \\ \|x\|_{\mathcal{C}_{\infty,1}^n} &= \max \Big\{ \|x\|_\mathcal{M}, \sqrt{n} \, \big\| \mathsf{E}(x^*x)^{\frac12} \big\|_\mathcal{N} \Big\}. \end{aligned}$$

Here the letters $\mathcal{R}$ and $\mathcal{C}$ stand for row and column according to Lemma 2.1. The symbol ∞ is motivated because in the following section we shall consider L_p versions of these spaces. The number 1 arises from interpolation theory, because we think of these spaces as endpoints in an interpolation scale. Finally, the norms on the right induce to introduce the spaces $L_\infty^r(\mathcal{M},\mathsf{E})$ and $L_\infty^c(\mathcal{M},\mathsf{E})$ as the closure of $\mathcal{M}$ with respect to the norms

$$\big\|\mathsf{E}(xx^*)^{\frac12}\big\|_{\mathcal{N}} \quad \text{and} \quad \big\|\mathsf{E}(x^*x)^{\frac12}\big\|_{\mathcal{N}}.$$

In this way, we obtain the spaces

$$\begin{array}{rcl}\mathcal{R}_{\infty,1}^n(\mathcal{M},\mathsf{E}) & = & \mathcal{M}\cap\sqrt{n}\,L_\infty^r(\mathcal{M},\mathsf{E}),\\ \mathcal{C}_{\infty,1}^n(\mathcal{M},\mathsf{E}) & = & \mathcal{M}\cap\sqrt{n}\,L_\infty^c(\mathcal{M},\mathsf{E}).\end{array}$$

Remark 2.2. It is easily seen that

$$\begin{array}{rcl}\big\|\mathsf{E}(xx^*)^{\frac12}\big\|_{\mathcal{N}} & = & \sup\Big\{\|\alpha x\|_{L_2(\mathcal{M})}\ \big|\ \|\alpha\|_{L_2(\mathcal{N})}\le 1\Big\},\\ \big\|\mathsf{E}(x^*x)^{\frac12}\big\|_{\mathcal{N}} & = & \sup\Big\{\|x\beta\|_{L_2(\mathcal{M})}\ \big|\ \|\beta\|_{L_2(\mathcal{N})}\le 1\Big\}\end{array}$$

This relation will be crucial in this paper and will be assumed in what follows.

The state φ induces the *n.f.* state $\phi=\varphi\circ\mathsf{E}_\mathcal{N}$ on $\mathcal{A}$. If $\mathcal{A}_{\oplus n}$ denotes the n-fold direct sum $\mathcal{A}\oplus\mathcal{A}\oplus\ldots\oplus\mathcal{A}$, we consider the *n.f.* state $\phi_n:\mathcal{A}_{\oplus n}\to\mathbb{C}$ and the conditional expectation $\mathcal{E}_n:\mathcal{A}_{\oplus n}\to\mathcal{A}$ given by

$$\phi_n\big(\sum_{k=1}^n a_k\otimes\delta_k\big)=\frac1n\sum_{k=1}^n\phi(a_k) \qquad \text{and} \qquad \mathcal{E}_n\big(\sum_{k=1}^n a_k\otimes\delta_k\big)=\frac1n\sum_{k=1}^n a_k.$$

Let us consider the map

$$u: x\in\mathcal{M}\mapsto\sum_{k=1}^n x_k\otimes\delta_k\in\mathcal{A}_{\oplus n} \quad \text{with} \quad x_k=\pi_k(x,-x). \tag{2.1}$$

Lemma 2.3. *The mappings*

$$\begin{array}{rrcl}u_r: & x\in\mathcal{R}_{\infty,1}^n(\mathcal{M},\mathsf{E}) & \mapsto & \displaystyle\sum_{k=1}^n x_k\otimes e_{1k}\in R_n(\mathcal{A}),\\ u_c: & x\in\mathcal{C}_{\infty,1}^n(\mathcal{M},\mathsf{E}) & \mapsto & \displaystyle\sum_{k=1}^n x_k\otimes e_{k1}\in C_n(\mathcal{A}),\end{array}$$

are isomorphisms onto complemented subspaces with constants independent of n.

Proof. Given $x\in\mathcal{R}_{\infty,1}^n(\mathcal{M},\mathsf{E})$, Lemma 2.1 gives

$$\big\|u_r(x)\big\|_{R_n(\mathcal{A})}=\Big\|\big(\sum_{k=1}^n x_kx_k^*\big)^{\frac12}\Big\|_{\mathcal{A}}\sim\max_{1\le k\le n}\|x_k\|_{\mathsf{A}_k}+\Big\|\Big(\sum_{k=1}^n\mathsf{E}_\mathcal{N}\big(x_kx_k^*\big)\Big)^{\frac12}\Big\|_{\mathcal{N}}.$$

In other words, we have

$$\big\|u_r(x)\big\|_{R_n(\mathcal{A})}\sim\|x\|_\mathcal{M}+\sqrt{n}\,\|x\|_{L_\infty^r(\mathcal{M},\mathsf{E})}\sim\|x\|_{\mathcal{R}_{\infty,1}^n(\mathcal{M},\mathsf{E})}.$$

Thus u_r is an isomorphism onto its image with constants independent of n. The same argument yields to the same conclusion for u_c. Let d_φ and d_ϕ be the densities

associated to the states φ and ϕ. To prove the complementation, we consider the map

$$\omega_r : x \in L_1(\mathcal{M}) + \frac{1}{\sqrt{n}}\, L_1^r(\mathcal{M},\mathsf{E}) \longmapsto \frac{1}{n}\sum_{k=1}^n x_k \otimes e_{1k} \in R_1^n(L_1(\mathcal{A})),$$

where $L_1^r(\mathcal{M},\mathsf{E})$ is the closure of $\mathcal{N} d_\varphi \mathcal{M}$ with respect to the norm $\|\mathsf{E}(xx^*)^{\frac12}\|_1$. Now we use approximation and assume that $x = \alpha d_\varphi a$ for some $(\alpha, a) \in \mathcal{N}\times\mathcal{M}$. Then, taking $a_k = \pi_k(a,-a)$ it follows from Theorem 7.1 in [19] that

$$\begin{aligned} \big\|\omega_r(x)\big\|_{R_1^n(L_1(\mathcal{A}))} &= \frac{1}{n}\Big\|\alpha d_\phi \Big(\sum_{k=1}^n a_k a_k^*\Big) d_\phi \alpha^*\Big\|_{L_{1/2}(\mathcal{A})}^{1/2} \\ &\le \frac{1}{n}\Big\|\alpha d_\varphi \Big(\sum_{k=1}^n \mathsf{E}_{\mathcal{N}}\big(a_k a_k^*\big)\Big) d_\varphi \alpha^*\Big\|_{L_{1/2}(\mathcal{N})}^{1/2}. \end{aligned}$$

This gives

$$\big\|\omega_r(x)\big\|_{R_1^n(L_1(\mathcal{A}))} \le \frac{1}{\sqrt{n}}\,\|x\|_{L_1^r(\mathcal{M},\mathsf{E})}.$$

On the other hand, by the triangle inequality

$$\big\|\omega_r(x)\big\|_{R_1^n(L_1(\mathcal{A}))} = \frac{1}{n}\Big\|\sum_{k=1}^n x_k\otimes e_{1k}\Big\|_{R_1^n(L_1(\mathcal{A}))} \le \|x\|_{L_1(\mathcal{M})}.$$

These estimates show that ω_r is a contraction. Note also that

$$\big\langle u_r(x), \omega_r(y)\big\rangle = \frac{1}{n}\sum_{k=1}^n \mathrm{tr}_{\mathcal{A}}\big(x_k^* y_k\big) = \frac{1}{n}\sum_{k=1}^n \mathrm{tr}_{\mathcal{M}}(x^*y) = \langle x, y\rangle.$$

In particular, since it follows from Corollary 2.12 of [10] that

$$\mathcal{R}_{\infty,1}^n(\mathcal{M};\mathsf{E}) = \Big(L_1(\mathcal{M}) + \frac{1}{\sqrt{n}}\, L_1^r(\mathcal{M},\mathsf{E})\Big)^*,$$

it turns out that the map $\omega_r^* u_r$ is the identity on $\mathcal{R}_{\infty,1}^n(\mathcal{M};\mathsf{E})$ and $u_r\omega_r^*$ is a bounded projection onto the image of u_r with constants independent of n. This completes the proof in the row case. The column case follows in the same way. □

In what follows we shall use the vector-valued space $L_\infty(\mathcal{A};\mathrm{OH}_n)$. This space is defined in [33] for $\mathcal{A}$ hyperfinite, but we shall work in this paper with $\mathcal{A}$ being a reduced free product von Neumann algebra as defined above, which is no longer hyperfinite. There is however a natural definition mentioned in the Introduction and motivated by Pisier's formula $\mathrm{OH}_n = [C_n, R_n]_{1/2}$. Indeed, recalling that the spaces $R_n(\mathcal{A})$ and $C_n(\mathcal{A})$ are defined for any von Neumann algebra $\mathcal{A}$, we may define

$$L_\infty(\mathcal{A};\mathrm{OH}_n) = \big[C_n(\mathcal{A}), R_n(\mathcal{A})\big]_{\frac12}. \tag{2.2}$$

Pisier showed in [31] that

$$\Big\|\sum_{k=1}^n a_k\otimes\delta_k\Big\|_{L_\infty(\mathcal{A};\mathrm{OH}_n)} = \sup\left\{\Big\|\sum_{k=1}^n a_k^*\alpha a_k\Big\|_{L_2(\mathcal{A})}^{\frac12} \mid \alpha\ge 0,\ \|\alpha\|_2\le 1\right\}. \tag{2.3}$$

More general results can be found in [16, 53] or Section 3 below.

Remark 2.4. We know from [10] that the spaces

$$L_\infty(\mathcal{A};\ell_1) \quad \text{and} \quad L_\infty(\mathcal{A};\ell_\infty)$$

are also defined for every von Neumann algebra $\mathcal{A}$. In particular, we might wonder whether or not our definition (2.2) of $L_\infty(\mathcal{A};\mathrm{OH}_n)$ satisfies the following complete isometry

$$L_\infty(\mathcal{A};\mathrm{OH}_n) = \big[L_\infty(\mathcal{A};\ell_\infty^n), L_\infty(\mathcal{A};\ell_1^n) \big]_{\frac12}.$$

Fortunately this is the case. A similar remark holds for $L_p(\mathcal{A};\mathrm{OH}_n)$, see [16].

The careful reader will have observed that the projection maps $u_r w_r^*$ and $u_c w_c^*$ are the same, modulo the identification of $R_n(\mathcal{A})$ and $C_n(\mathcal{A})$ with $\mathcal{A}_{\oplus n}$. This is the same identification as in Pisier's result (2.3). In particular, this allows us to identify via Lemma 2.3 the interpolation space

$$\mathrm{X}_{\frac12} = \big[\mathcal{C}_{\infty,1}^n(\mathcal{M},\mathsf{E}), \mathcal{R}_{\infty,1}^n(\mathcal{M},\mathsf{E}) \big]_{\frac12}$$

with a complemented subspace of $L_\infty(\mathcal{A};\mathrm{OH}_n)$. However, the difficult part in proving Theorem B is to identify the norm of the space $\mathrm{X}_{\frac12}$. Of course, according to the fact that we are interpolating 2-term intersection spaces, we expect a 4-term maximum. This is the case and we define $\mathcal{J}_{\infty,2}^n(\mathcal{M},\mathsf{E})$ as the space of elements x in $\mathcal{M}$ equipped with the norm

$$\max_{u,v\in\{4,\infty\}} \Big\{ n^{\frac{1}{\xi(u,v)}} \sup \big\{ \|\alpha x \beta\|_{L_{\xi(u,v)}(\mathcal{M})} \, \big| \, \|\alpha\|_{L_u(\mathcal{N})}, \|\beta\|_{L_v(\mathcal{N})} \le 1 \big\} \Big\},$$

where $\xi(u,v)$ is given by $\frac{1}{\xi(u,v)} = \frac1u + \frac1v$. Obviously, multiplying by elements α,β in the unit ball of $L_\infty(\mathcal{N})$ and taking suprema does not contribute to the corresponding $L_{\xi(u,v)}(\mathcal{M})$ term. In other words, we may rewrite the norm of x in $\mathcal{J}_{\infty,2}^n(\mathcal{M},\mathsf{E})$ as

$$\|x\|_{\mathcal{J}_{\infty,2}^n(\mathcal{M},\mathsf{E})} = \max \Big\{ \|x\|_{\Lambda_{(u,v)}^n} \, \Big| \, u,v \in \{4,\infty\} \Big\}$$

where the $\Lambda_{(u,v)}^n$ norms are given by

$$\begin{aligned}
\|x\|_{\Lambda_{(\infty,\infty)}^n} &= \|x\|_{\mathcal{M}}, \\
\|x\|_{\Lambda_{(\infty,4)}^n} &= n^{\frac14} \sup \Big\{ \|x\beta\|_{L_4(\mathcal{M})} \, \Big| \, \|\beta\|_{L_4(\mathcal{N})} \le 1 \Big\}, \\
\|x\|_{\Lambda_{(4,\infty)}^n} &= n^{\frac14} \sup \Big\{ \|\alpha x\|_{L_4(\mathcal{M})} \, \Big| \, \|\alpha\|_{L_4(\mathcal{N})} \le 1 \Big\}, \\
\|x\|_{\Lambda_{(4,4)}^n} &= n^{\frac12} \sup \Big\{ \|\alpha x\beta\|_{L_2(\mathcal{M})} \, \Big| \, \|\alpha\|_{L_4(\mathcal{N})}, \|\beta\|_{L_4(\mathcal{N})} \le 1 \Big\}.
\end{aligned}$$

These norms arise as particular cases of the so-called conditional L_p spaces, to be analyzed in Section 3. Before identifying the norm of $\mathrm{X}_{\frac12}$, we need some information on interpolation spaces.

Lemma 2.5. *If $(1/u,1/v) = (\theta/2,(1-\theta)/2)$, we have for $x \in \mathcal{M}$*

$$\begin{aligned}
\|x\|_{[\mathcal{M},L_\infty^r(\mathcal{M},\mathsf{E})]_\theta} &= \sup \Big\{ \|\alpha x\|_{L_u(\mathcal{M})} \, \Big| \, \|\alpha\|_{L_u(\mathcal{N})} \le 1 \Big\}, \\
\|x\|_{[L_\infty^c(\mathcal{M},\mathsf{E}),\mathcal{M}]_\theta} &= \sup \Big\{ \|x\beta\|_{L_v(\mathcal{M})} \, \Big| \, \|\beta\|_{L_v(\mathcal{N})} \le 1 \Big\}, \\
\|x\|_{[L_\infty^c(\mathcal{M},\mathsf{E})]_\theta, L_\infty^r(\mathcal{M},\mathsf{E})]_\theta} &= \sup \Big\{ \|\alpha x\beta\|_{L_2(\mathcal{M})} \, \Big| \, \|\alpha\|_{L_u(\mathcal{N})}, \|\beta\|_{L_v(\mathcal{N})} \le 1 \Big\}.
\end{aligned}$$

The proof can be found in [16]. In the finite setting, this result follows from a well-known application of Helson/Lowdenslager, Wiener/Masani type results on the existence of operator-valued analytic functions. This kind of applications has been used extensively by Pisier in his theory of vector-valued L_p spaces.

Theorem 2.6. *We have isomorphically*

$$\big[\mathcal{C}_{\infty,1}^n(\mathcal{M},\mathsf{E}),\mathcal{R}_{\infty,1}^n(\mathcal{M},\mathsf{E})\big]_{\frac{1}{2}} \quad \simeq \quad \mathcal{J}_{\infty,2}^n(\mathcal{M},\mathsf{E}).$$

Moreover, the constants in these isomorphisms are uniformly bounded on n.

Proof. We have a contractive inclusion

$$\mathrm{X}_{\frac{1}{2}} \subset \mathcal{J}_{\infty,2}^n(\mathcal{M},\mathsf{E}).$$

Indeed, by elementary properties of interpolation spaces we find

$$\mathrm{X}_{\frac{1}{2}} \subset [\mathcal{M},\mathcal{M}]_{\frac{1}{2}} \cap [\sqrt{n}L_\infty^c,\mathcal{M}]_{\frac{1}{2}} \cap [\mathcal{M},\sqrt{n}L_\infty^r]_{\frac{1}{2}} \cap [\sqrt{n}L_\infty^c,\sqrt{n}L_\infty^r]_{\frac{1}{2}},$$

where L_∞^r and L_∞^c are abbreviations for $L_\infty^r(\mathcal{M},\mathsf{E})$ and $L_\infty^c(\mathcal{M},\mathsf{E})$ respectively. Using the obvious identity $[\lambda_0\mathrm{X}_0,\lambda_1\mathrm{X}_1]_\theta = \lambda_0^{1-\theta}\lambda_1^\theta\,\mathrm{X}_\theta$ and applying Lemma 2.5 we rediscover the norm of the space $\mathcal{J}_{\infty,2}^n(\mathcal{M},\mathsf{E})$ on the right hand side. Therefore the lower estimate holds with constant 1.

To prove the upper estimate, we note from Lemma 2.3 and (2.3) that

$$\begin{aligned}
\|x\|_{\mathrm{X}_{\frac{1}{2}}} &\sim \big\|u(x)\big\|_{[C_n(\mathcal{A}),R_n(\mathcal{A})]_{\frac{1}{2}}} \\
&= \Big\|\sum_{k=1}^n x_k\otimes\delta_k\Big\|_{L_\infty(\mathcal{A};\mathrm{OH}_n)} \\
&= \sup\Big\{\Big\|\sum_{k=1}^n x_k^*ax_k\Big\|_{L_2(\mathcal{A})}^{\frac{1}{2}} \ \big| \ a\ge 0,\ \|a\|_2\le 1\Big\} = \mathrm{A}.
\end{aligned}$$

Thus, it remains to see that

$$\mathrm{A} \ \lesssim \max\Big\{\|x\|_{\Lambda_{(u,v)}^n} \ \big| \ u,v\in\{4,\infty\}\Big\} = \|x\|_{\mathcal{J}_{\infty,2}^n(\mathcal{M},\mathsf{E})}. \tag{2.4}$$

In order to justify (2.4), we introduce the orthogonal projections L_k and R_k in $L_2(\mathcal{A})$ defined as follows. Given $a\in L_2(\mathcal{A})$, the vector $\mathsf{L}_k(a)$ (resp. $\mathsf{R}_k(a)$) collects the reduced words in a starting (resp. ending) with a letter in A_k. In other words, following standard terminology in free probability, we have

$$\begin{aligned}
\mathsf{L}_k : L_2(\mathcal{A}) &\longrightarrow L_2\Big(\big[\bigoplus_{m\ge1}\bigoplus_{j_1=k\neq j_2\neq\cdots\neq j_m}\mathring{\mathsf{A}}_{j_1}\mathring{\mathsf{A}}_{j_2}\cdots\mathring{\mathsf{A}}_{j_m}\big]''\Big), \\
\mathsf{R}_k : L_2(\mathcal{A}) &\longrightarrow L_2\Big(\big[\bigoplus_{m\ge1}\bigoplus_{j_1\neq j_2\neq\cdots\neq k=j_m}\mathring{\mathsf{A}}_{j_1}\mathring{\mathsf{A}}_{j_2}\cdots\mathring{\mathsf{A}}_{j_m}\big]''\Big).
\end{aligned}$$

Now, given a positive operator a in $L_2(\mathcal{A})$ and a fixed integer $1\le k\le n$, we consider the following way to decompose a in terms of the projections L_k and R_k and the conditional expectation $\mathsf{E}_\mathcal{N}:\mathcal{A}\to\mathcal{N}$

$$a = \mathsf{E}_\mathcal{N}(a) + \mathsf{L}_k(a) + \mathsf{R}_k(a) - \mathsf{R}_k\mathsf{L}_k(a) + \gamma_k(a), \tag{2.5}$$

where the term $\gamma_k(a)$ has the following form

$$\gamma_k(a) = a - \mathsf{E}_\mathcal{N}(a) - \mathsf{L}_k(a) - \mathsf{R}_k(a-\mathsf{L}_k(a)).$$

The triangle inequality gives $\mathrm{A}^2 \le \sum_{k=1}^5 \mathrm{A}_j^2$, where the terms A_j are the result of replacing a in A by the j-th term in the decomposition (2.5). Let us estimate these terms separately. For the first term $\mathsf{E}_{\mathcal{N}}(a)$ we use

$$\begin{aligned}
\mathrm{A}_1^2 & = \Big\| \sum_{k=1}^n x_k^* \mathsf{E}_{\mathcal{N}}(a) x_k \Big\|_{L_2(\mathcal{A})} \\
& \le \Big\| \sum_{k=1}^n \mathsf{E}_{\mathcal{N}} \big(x_k^* \mathsf{E}_{\mathcal{N}}(a) x_k \big) \Big\|_{L_2(\mathcal{N})} \\
& + \Big\| \sum_{k=1}^n x_k^* \mathsf{E}_{\mathcal{N}}(a) x_k - \mathsf{E}_{\mathcal{N}} \big(x_k^* \mathsf{E}_{\mathcal{N}}(a) x_k \big) \Big\|_{L_2(\mathcal{A})} = \mathrm{A}_{11}^2 + \mathrm{A}_{12}^2.
\end{aligned}$$

Since $\mathsf{E}_{\mathcal{N}}\big(x_k^* \mathsf{E}_{\mathcal{N}}(a) x_k\big) = \mathsf{E}\big(x^* \mathsf{E}_{\mathcal{N}}(a) x\big)$ and $a \in \mathsf{B}_{L_2(\mathcal{A})}^+$, we obtain

$$\begin{aligned}
\mathrm{A}_{11} & = n^{\frac12} \sup \Big\{ \mathrm{tr}_{\mathcal{N}} \Big(\beta^* \mathsf{E}\big(x^* \mathsf{E}_{\mathcal{N}}(a) x\big) \beta \Big)^{\frac12} \ \big| \ \|\beta\|_{L_4(\mathcal{N})} \le 1 \Big\} \\
& \le n^{\frac12} \sup \Big\{ \mathrm{tr}_{\mathcal{M}} \big(\beta^* x^* \alpha^* \alpha x \beta \big)^{\frac12} \ \big| \ \|\alpha\|_{L_4(\mathcal{N})}, \|\beta\|_{L_4(\mathcal{N})} \le 1 \Big\}.
\end{aligned}$$

This gives $\mathrm{A}_{11} \le \|x\|_{\Lambda_{(4,4)}^n} \le \|x\|_{\mathcal{J}_{\infty,2}^n(\mathcal{M},\mathsf{E})}$. On the other hand, by freeness

$$\begin{aligned}
\mathrm{A}_{12}^2 & = \Big(\sum_{k=1}^n \big\| x_k^* \mathsf{E}_{\mathcal{N}}(a) x_k - \mathsf{E}_{\mathcal{N}} \big(x_k^* \mathsf{E}_{\mathcal{N}}(a) x_k \big) \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac12} \\
& \le 2 \Big(\sum_{k=1}^n \big\| x_k^* \mathsf{E}_{\mathcal{N}}(a) x_k \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac12} = 2\, n^{\frac12} \, \big\| x^* \mathsf{E}_{\mathcal{N}}(a) x \big\|_{L_2(\mathcal{M})}.
\end{aligned}$$

Then positivity gives

$$\mathrm{A}_{12} \le \sqrt{2}\, \|x\|_{\Lambda_{(4,\infty)}^n} \le \sqrt{2}\, \|x\|_{\mathcal{J}_{\infty,2}^n(\mathcal{M},\mathsf{E})}.$$

The second, third and fourth terms in (2.5) satisfy

$$\begin{aligned}
& \Big\| \sum_{k=1}^n x_k^* \big(\mathsf{L}_k(a) + \mathsf{R}_k(a) + \mathsf{R}_k \mathsf{L}_k(a) \big) x_k \Big\|_{L_2(\mathcal{A})} \\
& = \sup \Big\{ \sum_{k=1}^n \mathrm{tr}_{\mathcal{A}} \Big(b x_k^* \big(\mathsf{L}_k(a) + \mathsf{R}_k(a) + \mathsf{R}_k \mathsf{L}_k(a) \big) x_k \Big) \ \big| \ \|b\|_{L_2(\mathcal{A})} \le 1 \Big\} \\
& = \sup \Big\{ \sum_{k=1}^n \mathrm{tr}_{\mathcal{A}} \Big(x_k b x_k^* \big(\mathsf{L}_k(a) + \mathsf{R}_k(a) + \mathsf{R}_k \mathsf{L}_k(a) \big) \Big) \ \big| \ \|b\|_{L_2(\mathcal{A})} \le 1 \Big\} \\
& \le \sup_{\|b\|_{L_2(\mathcal{A})} \le 1} \Big(\sum_{k=1}^n \big\| x_k b x_k^* \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac12} \Big(\sum_{k=1}^n \big\| \mathsf{L}_k(a) + \mathsf{R}_k(a) + \mathsf{R}_k \mathsf{L}_k(a) \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac12}.
\end{aligned}$$

The second factor on the right is estimated by orthogonality

$$\Big(\sum_{k=1}^n \big\| \mathsf{L}_k(a) + \mathsf{R}_k(a) + \mathsf{R}_k \mathsf{L}_k(a) \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac12} \le 3\, \|a\|_{L_2(\mathcal{A})} \le 3.$$

For the first factor, we write b as a linear combination $(b_1 - b_2) + i(b_3 - b_4)$ of four positive operators. Therefore, all these terms are covered by the following estimate, to be proved below.

Claim. Given $a \in \mathsf{B}^+_{L_2(\mathcal{A})}$, we have

$$\Big(\sum_{k=1}^n \big\| x_k a x_k^* \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac14} \lesssim \max \Big\{ \|x\|_{\mathcal{M}}, \, n^{\frac14} \sup_{\|\beta\|_{L_4(\mathcal{N})} \le 1} \|x\beta\|_{L_4(\mathcal{M})} \Big\}. \tag{2.6}$$

Before justifying our claim, we complete the proof. It remains to estimate the term A_5 associated to $\gamma_k(a)$. We first observe that $\gamma_k(a)$ is a mean-zero element of $L_2(\mathcal{A})$ made up of reduced words not starting nor ending with a letter in A_k. Indeed, note that $\mathsf{E}_{\mathcal{N}}(\gamma_k(a)) = 0$ and that we first eliminate the words starting with a letter in A_k by subtracting $\mathsf{L}_k(a)$ and, after that, we eliminate the remaining words which end with a letter in A_k by subtracting $\mathsf{R}_k(a - \mathsf{L}_k(a))$. Therefore, it turns out that $x_1^*\gamma_1(a)x_1, x_2^*\gamma_2(a)x_2, \ldots, x_n^*\gamma_n(a)x_n$ is a free family of random variables. In particular, by orthogonality

$$\Big\| \sum_{k=1}^n x_k^* \gamma_k(a) x_k \Big\|_{L_2(\mathcal{A})}^{\frac12} \quad = \quad \Big(\sum_{k=1}^n \big\| x_k^* \gamma_k(a) x_k \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac14}.$$

However, recalling that $\gamma_k(a)$ is a mean-zero element made up of words not starting nor ending with a letter in A_k, the following identities hold for the conditional expectation $\mathcal{E}_{\mathsf{A}_k} : L_2(\mathcal{A}) \to L_2(\mathsf{A}_k)$

$$\begin{aligned} \mathcal{E}_{\mathsf{A}_k} \Big(\gamma_k(a)^* \big(x_k x_k^* - \mathsf{E}_{\mathcal{N}}(x_k x_k^*) \big) \gamma_k(a) \Big) &= 0, \\ \mathcal{E}_{\mathsf{A}_k} \Big(\gamma_k(a) \big(x_k x_k^* - \mathsf{E}_{\mathcal{N}}(x_k x_k^*) \big) \gamma_k(a)^* \Big) &= 0. \end{aligned} \tag{2.7}$$

Using property (2.7) we find

$$\begin{aligned} \big\| x_k^* \gamma_k(a) x_k \big\|_{L_2(\mathcal{A})}^2 & = & \mathrm{tr}_{\mathcal{A}} \big(x_k^* \gamma_k(a)^* x_k x_k^* \gamma_k(a) x_k \big) \\ & = & \mathrm{tr}_{\mathcal{A}} \big(x_k^* \mathcal{E}_{\mathsf{A}_k} \big(\gamma_k(a)^* x_k x_k^* \gamma_k(a) \big) x_k \big) \\ & = & \mathrm{tr}_{\mathcal{A}} \big(x_k^* \mathcal{E}_{\mathsf{A}_k} \big(\gamma_k(a)^* \mathsf{E}_{\mathcal{N}}(x_k x_k^*) \gamma_k(a) \big) x_k \big) \\ & = & \mathrm{tr}_{\mathcal{A}} \big(\gamma_k(a) x_k x_k^* \gamma_k(a)^* \mathsf{E}_{\mathcal{N}}(x_k x_k^*) \big) \\ & = & \mathrm{tr}_{\mathcal{A}} \big(\mathcal{E}_{\mathsf{A}_k} \big(\gamma_k(a) x_k x_k^* \gamma_k(a)^* \big) \mathsf{E}_{\mathcal{N}}(x_k x_k^*) \big) \\ & = & \mathrm{tr}_{\mathcal{A}} \big(\mathcal{E}_{\mathsf{A}_k} \big(\gamma_k(a) \mathsf{E}_{\mathcal{N}}(x_k x_k^*) \gamma_k(a)^* \big) \mathsf{E}_{\mathcal{N}}(x_k x_k^*) \big) \\ & = & \mathrm{tr}_{\mathcal{A}} \big(\mathsf{E}_{\mathcal{N}}(x_k x_k^*)^{\frac12} \gamma_k(a) \mathsf{E}_{\mathcal{N}}(x_k x_k^*) \gamma_k(a)^* \mathsf{E}_{\mathcal{N}}(x_k x_k^*)^{\frac12} \big). \end{aligned}$$

In combination with $\|\gamma_k(a)\|_2 \le 5 \, \|a\|_2$ and Hölder's inequality, this yields

$$\big\| x_k^* \gamma_k(a) x_k \big\|_{L_2(\mathcal{A})}^2 = \big\| \mathsf{E}_{\mathcal{N}}(xx^*)^{\frac12} \gamma_k(a) \mathsf{E}_{\mathcal{N}}(xx^*)^{\frac12} \big\|_{L_2(\mathcal{A})}^2 \le 25 \, \|x\|_{L_\infty^r(\mathcal{M},\mathsf{E})}^4.$$

We refer to Lemma 2.5 or [16] for the fact that

$$\|x\|_{L_\infty^r(\mathcal{M},\mathsf{E})} \le \sup \Big\{ \|\alpha x\|_{L_4(\mathcal{M})} \ \big| \ \|\alpha\|_{L_4(\mathcal{N})} \le 1 \Big\}.$$

The inequalities proved so far give rise to the following estimate

$$\Big\| \sum_{k=1}^n x_k^* \gamma_k(a) x_k \Big\|_{L_2(\mathcal{A})}^{\frac12} \le \sqrt{5} \, \|x\|_{\Lambda_{(4,\infty)}^n} \le \sqrt{5} \, \|x\|_{\mathcal{J}_{\infty,2}^n(\mathcal{M},\mathsf{E})}.$$

Therefore, it remains to prove the claim. We proceed in a similar way. According to the decomposition (2.5), we may use the triangle inequality and decompose the

left hand side of (2.6) into five terms $\mathrm{B}_1, \mathrm{B}_2, \ldots, \mathrm{B}_5$. For the first term, we deduce from positivity that

$$\Big(\sum_{k=1}^n \big\| x_k \mathsf{E}_\mathcal{N}(a) x_k^* \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac14} = n^{\frac14} \big\| x \mathsf{E}_\mathcal{N}(a) x^* \big\|_{L_2(\mathcal{M})}^{\frac12} \le n^{\frac14} \sup_{\|\beta\|_{L_4(\mathcal{N})} \le 1} \|x\beta\|_{L_4(\mathcal{M})}.$$

The terms $\mathrm{B}_2, \mathrm{B}_3$ and B_4 satisfy

$$\begin{aligned} \Big(\sum_{k=1}^n \big\| x_k^* \mathsf{L}_k(a) x_k \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac14} & \le \|x\|_\mathcal{M} \Big(\sum_{k=1}^n \|\mathsf{L}_k(a)\|_{L_2(\mathcal{A})}^2 \Big)^{\frac14}, \\ \Big(\sum_{k=1}^n \big\| x_k^* \mathsf{R}_k(a) x_k \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac14} & \le \|x\|_\mathcal{M} \Big(\sum_{k=1}^n \|\mathsf{R}_k(a)\|_{L_2(\mathcal{A})}^2 \Big)^{\frac14}, \\ \Big(\sum_{k=1}^n \big\| x_k^* \mathsf{R}_k \mathsf{L}_k(a) x_k \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac14} & \le \|x\|_\mathcal{M} \Big(\sum_{k=1}^n \|\mathsf{R}_k \mathsf{L}_k(a)\|_{L_2(\mathcal{A})}^2 \Big)^{\frac14}. \end{aligned}$$

Therefore, by orthogonality we have

$$\Big(\sum_{k=1}^n \big\| x_k^* \big(\mathsf{L}_k(a) + \mathsf{R}_k(a) + \mathsf{R}_k \mathsf{L}_k(a) \big) x_k \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac14} \le 3 \, \|x\|_\mathcal{M}.$$

This leaves us with the term B_5. Arguing as above

$$\Big(\sum_{k=1}^n \big\| x_k \gamma_k(a) x_k^* \big\|_{L_2(\mathcal{A})}^2 \Big)^{\frac14} \le \sqrt{5} \; n^{\frac14} \sup_{\|\beta\|_{L_4(\mathcal{N})} \le 1} \|x\beta\|_{L_4(\mathcal{M})}.$$

Therefore, the claim holds and the proof is complete. □

Remark 2.7. The arguments in Theorem 2.6 also give

$$\begin{aligned} \|x\|_{[\mathcal{M}, \mathcal{R}_{\infty,1}^n(\mathcal{M},\mathsf{E})]_{\frac12}} & \sim \max \Big\{ \|x\|_\mathcal{M}, \|x\|_{\Lambda_{(4,\infty)}^n} \Big\}, \\ \|x\|_{[\mathcal{C}_{\infty,1}^n(\mathcal{M},\mathsf{E}) \, , \, \mathcal{M}]_{\frac12}} & \sim \max \Big\{ \|x\|_\mathcal{M}, \|x\|_{\Lambda_{(\infty,4)}^n} \Big\}. \end{aligned}$$

Now we show how the space $\mathrm{X}_{1/2}$ is related to Theorem B. The idea follows from a well-known argument in which complete boundedness arise as a particular case of amalgamation. More precisely, if $L_2^r(\mathcal{M}) / L_2^c(\mathcal{M})$ denote the row/column quantizations of $L_2(\mathcal{M})$ and $2 \le q \le \infty$, the row/column operator space structures on $L_q(\mathcal{M})$ are defined as follows

$$\begin{aligned} L_q^r(\mathcal{M}) & = \big[\mathcal{M}, L_2^r(\mathcal{M}) \big]_{\frac2q}, \\ L_q^c(\mathcal{M}) & = \big[\mathcal{M}, L_2^c(\mathcal{M}) \big]_{\frac2q}. \end{aligned} \tag{2.8}$$

The following result from [16] is a generalized form of (3).

Lemma 2.8. *If $\mathcal{M}_m = \mathrm{M}_m(\mathcal{M})$, we have*

$$\begin{aligned} \big\| d_\varphi^{\frac14} (x_{ij}) \big\|_{\mathrm{M}_m(L_4^r(\mathcal{M}))} & = \sup_{\|\alpha\|_{S_4^m} \le 1} \Big\| d_\varphi^{\frac14} \Big(\sum_{k=1}^m \alpha_{ik} x_{kj} \Big) \Big\|_{L_4(\mathcal{M}_m)}, \\ \big\| (x_{ij}) d_\varphi^{\frac14} \big\|_{\mathrm{M}_m(L_4^c(\mathcal{M}))} & = \sup_{\|\beta\|_{S_4^m} \le 1} \Big\| \Big(\sum_{k=1}^m x_{ik} \beta_{kj} \Big) d_\varphi^{\frac14} \Big\|_{L_4(\mathcal{M}_m)}. \end{aligned}$$

The proof follows from

$$\begin{aligned}\big\|d_\varphi^{\frac12}(x_{ij})\big\|_{\mathrm{M}_m(L_2^r(\mathcal{M}))} &= \sup_{\|\alpha\|_{S_2^m}\le 1} \Big\|d_\varphi^{\frac12}\Big(\sum_{k=1}^m \alpha_{ik}x_{kj}\Big)\Big\|_{L_2(\mathcal{M}_m)},\\ \big\|(x_{ij})d_\varphi^{\frac12}\big\|_{\mathrm{M}_m(L_2^c(\mathcal{M}))} &= \sup_{\|\beta\|_{S_2^m}\le 1} \Big\|\Big(\sum_{k=1}^m x_{ik}\beta_{kj}\Big)d_\varphi^{\frac12}\Big\|_{L_2(\mathcal{M}_m)},\end{aligned} \tag{2.9}$$

and some complex interpolation formulas developed in [16]. The identity (2.9) from which we interpolate is a well-known expression in operator space theory (see e.g. p.56 in [4]) and will play a crucial role in the last section of this paper. Now we define the space $\mathcal{J}_{\infty,2}^n(\mathcal{M})$ as follows

$$\mathcal{J}_{\infty,2}^n(\mathcal{M}) = \mathcal{M} \cap n^{\frac14} L_4^c(\mathcal{M}) \cap n^{\frac14} L_4^r(\mathcal{M}) \cap n^{\frac12} L_2(\mathcal{M}).$$

Lemma 2.8 determines the operator space structure of the cross terms in $\mathcal{J}_{\infty,2}^n(\mathcal{M})$. On the other hand, according to Pisier's fundamental identities (1.2) or (2.3), it is easily seen that we have

$$\big\|d_\varphi^{\frac14}(x_{ij})d_\varphi^{\frac14}\big\|_{\mathrm{M}_m(L_2(\mathcal{M}))} = \sup_{\|\alpha\|_{S_4^m},\|\beta\|_{S_4^m}\le 1} \Big\|d_\varphi^{\frac14}\Big(\sum_{k,l=1}^m \alpha_{ik}x_{kl}\beta_{lj}\Big)d_\varphi^{\frac14}\Big\|_{L_2(\mathcal{M}_m)}.$$

In other words, the o.s.s. of $\mathcal{J}_{\infty,2}^n(\mathcal{M})$ is described by the isometry

$$\mathrm{M}_m\big(\mathcal{J}_{\infty,2}^n(\mathcal{M})\big) = \mathcal{J}_{\infty,2}^n(\mathcal{M}_m, \mathsf{E}_m), \tag{2.10}$$

where $\mathcal{M}_m = \mathrm{M}_m(\mathcal{M})$ and $\mathsf{E}_m = id_{\mathrm{M}_m} \otimes \varphi : \mathcal{M}_m \to \mathrm{M}_m$ for $m \ge 1$. This means that the *vector-valued* spaces $\mathcal{J}_{\infty,2}^n(\mathcal{M}, \mathsf{E})$ describe the o.s.s. of the *scalar-valued* spaces $\mathcal{J}_{\infty,2}^n(\mathcal{M})$. In the result below we prove the operator space/free analogue of a form of Rosenthal's inequality in the limit case $p \to \infty$, see Section 3 or [16] for more details. This result does not have a commutative counterpart. The particular case for $\mathcal{M} = \mathcal{B}(\ell_2^n)$ recovers Theorem B. Given a von Neumann algebra $\mathcal{M}$, we set as above $\mathsf{A}_k = \mathcal{M} \oplus \mathcal{M}$.

Corollary 2.9. *If $\mathcal{A}_\mathcal{N} = *_\mathcal{N} \mathsf{A}_k$, the map*

$$u : x \in \mathcal{J}_{\infty,2}^n(\mathcal{M}, \mathsf{E}) \mapsto \sum_{k=1}^n x_k \otimes \delta_k \in L_\infty(\mathcal{A}_\mathcal{N}; \mathrm{OH}_n)$$

is an isomorphism with complemented image and constants independent of n. In particular, replacing as usual $(\mathcal{M}, \mathcal{N}, \mathsf{E})$ by $(\mathcal{M}_m, \mathrm{M}_m, \mathsf{E}_m)$ and replacing $\mathcal{A}_\mathcal{N}$ by the non-amalgamated algebra $\mathcal{A}_\mathbb{C} = \mathsf{A}_1 * \mathsf{A}_2 * \cdots * \mathsf{A}_n$, we obtain a cb-isomorphism with cb-complemented image and constants independent of n

$$\sigma : x \in \mathcal{J}_{\infty,2}^n(\mathcal{M}) \mapsto \sum_{k=1}^n x_k \otimes \delta_k \in L_\infty(\mathcal{A}_\mathbb{C}; \mathrm{OH}_n). \tag{$\Sigma_{\infty 2}$}$$

Proof. The first assertion follows from Lemma 2.3 and Theorem 2.6. To prove the second assertion we choose the triple $(\mathcal{M}_m, \mathrm{M}_m, \mathsf{E}_m)$ and apply (2.10). This provides us with an isomorphic embedding

$$\sigma_m : x \in \mathrm{M}_m\big(\mathcal{J}_{\infty,2}^n(\mathcal{M})\big) \mapsto \sum_{k=1}^n x_k \otimes \delta_k \in L_\infty(\mathcal{A}_m; \mathrm{OH}_n),$$

where the von Neumann algebra $\mathcal{A}_m$ is given by

$$\mathcal{A}_m = \mathrm{M}_m(\mathcal{A}_\mathbb{C}) = \mathrm{M}_m(\mathsf{A}_1) *_{\mathrm{M}_m} \mathrm{M}_m(\mathsf{A}_2) *_{\mathrm{M}_m} \cdots *_{\mathrm{M}_m} \mathrm{M}_m(\mathsf{A}_n).$$

The last isometry is well-known, see e.g. [11]. In particular

$$L_\infty(\mathcal{A}_m; \mathrm{OH}_n) = \mathrm{M}_m\big(L_\infty(\mathcal{A}_\mathbb{C}; \mathrm{OH}_n)\big)$$

and it turns out that $\sigma_m = id_{\mathrm{M}_m} \otimes \sigma$. This completes the proof. $\square$

Remark 2.10. A quick look at Corollary 2.9 shows that our formulation of $(\Sigma_{\infty 2})$ is the half-way result (in the sense of complex interpolation) between the stated isomorphisms in Lemma 2.1. In the same way, as we shall see in Section 3, the free analogue of (Σ_{p2}) is the half-way result between the row and column formulations of the free Rosenthal inequality [18] for positive random variables. However, this nice property is no longer true for (Σ_{pq}) with $q \neq 2$, see below for details.

2.2. **Embedding S_q into $L_1(\mathcal{A})$.** The tools developed so far allow us to prove Theorem C and thereby obtain a complete embedding of the Schatten class S_q into $L_1(\mathcal{A})$ for some QWEP von Neumann algebra $\mathcal{A}$. Our main concern here is to set a model from which we may motivate/justify the forthcoming definitions and arguments. We shall use some well-known facts from the theory of operator spaces which we do not state here to simplify the exposition. All these results will be properly stated in Section 4 and we shall refer to them. We fix $\mathcal{M} = \mathcal{B}(\ell_2)$ and consider a family $\gamma_1, \gamma_2, \ldots \in \mathbb{R}_+$ of strictly positive numbers. Then we define d_γ to be the diagonal operator on ℓ_2 defined by $\mathsf{d}_\gamma = \sum_k \gamma_k e_{kk}$. This operator can be regarded as the density d_ψ associated to a normal strictly semifinite faithful (*n.s.s.f.* in short) weight ψ on $\mathcal{B}(\ell_2)$. Let us set q_n to be the projection $\sum_{k \le n} e_{kk}$ and let us consider the restriction of ψ to the subalgebra $q_n \mathcal{B}(\ell_2) q_n$

$$\psi_n \Big(q_n \big(\sum\nolimits_{ij} x_{ij} e_{ij} \big) q_n \Big) = \sum_{k=1}^n \gamma_k x_{kk}.$$

Note that if we set $\mathrm{k}_n = \psi_n(q_n)$, we obtain $\psi_n = \mathrm{k}_n \varphi_n$ for some state φ_n on $q_n \mathcal{B}(\ell_2) q_n$. If d_{ψ_n} denotes the density on $q_n \mathcal{B}(\ell_2) q_n$ associated to the weight ψ_n, we define the space $\mathcal{J}_{\infty,2}(\psi_n)$ as the subspace

$$\Big\{ \big(z, z d_{\psi_n}^{\frac14}, d_{\psi_n}^{\frac14} z, d_{\psi_n}^{\frac14} z d_{\psi_n}^{\frac14} \big) \ \big| \ z \in q_n \mathcal{B}(\ell_2) q_n \Big\}$$

of the direct sum

$$\mathcal{L}_\infty^n = \big(C_n \otimes_h R_n \big) \oplus_2 \big(C_n \otimes_h \mathrm{OH}_n \big) \oplus_2 \big(\mathrm{OH}_n \otimes_h R_n \big) \oplus_2 \big(\mathrm{OH}_n \otimes_h \mathrm{OH}_n \big).$$

In other words, we obtain the intersection space considered in the Introduction

$$(C_n \otimes_h R_n) \cap (C_n \otimes_h \mathrm{OH}_n) d_{\psi_n}^{\frac14} \cap d_{\psi_n}^{\frac14} (\mathrm{OH}_n \otimes_h R_n) \cap d_{\psi_n}^{\frac14} (\mathrm{OH}_n \otimes_h \mathrm{OH}_n) d_{\psi_n}^{\frac14}.$$

Lemma 2.11. *Let us consider*

$$\mathcal{K}_{1,2}(\psi_n) = \mathcal{J}_{\infty,2}(\psi_n)^*.$$

Assume that $\mathrm{k}_n = \sum_{k=1}^n \gamma_k$ is an integer and define $\mathcal{A}_n$ to be the k_n-fold reduced free product of $q_n \mathcal{B}(\ell_2) q_n \oplus q_n \mathcal{B}(\ell_2) q_n$. If $\pi_j : q_n \mathcal{B}(\ell_2) q_n \oplus q_n \mathcal{B}(\ell_2) q_n \to \mathcal{A}_n$ is the natural embedding into the j-th component of $\mathcal{A}_n$ and we set $x_j = \pi_j(x, -x)$, the mapping

$$\omega : x \in \mathcal{K}_{1,2}(\psi_n) \mapsto \frac{1}{\mathrm{k}_n} \sum_{j=1}^{\mathrm{k}_n} x_j \otimes \delta_j \in L_1(\mathcal{A}_n; \mathrm{OH}_{\mathrm{k}_n})$$

is a cb-embedding with cb-complemented image and constants independent of n.

Proof. We claim that

$$\mathcal{J}_{\infty,2}(\psi_n) = \mathcal{J}_{\infty,2}^{\mathrm{k}_n}(q_n \mathcal{B}(\ell_2) q_n)$$

completely isometrically. Indeed, by (2.8)

$$\begin{aligned} \mathrm{k}_n^{\frac14} L_4^r(q_n \mathcal{B}(\ell_2) q_n, \varphi_n) &= \mathrm{k}_n^{\frac14} \big[\mathcal{B}(\ell_2^n), L_2^r(\mathcal{B}(\ell_2^n), \varphi_n) \big]_{\frac12} \\ &= \mathrm{k}_n^{\frac14} \big[\mathcal{B}(\ell_2^n), d_{\varphi_n}^{\frac12} L_2^r(\mathcal{B}(\ell_2^n), \mathrm{tr}_n) \big]_{\frac12} \\ &= \mathrm{k}_n^{\frac14} d_{\varphi_n}^{\frac14} \big[C_n \otimes_h R_n, R_n \otimes_h R_n \big]_{\frac12} = d_{\psi_n}^{\frac14} (\mathrm{OH}_n \otimes_h R_n). \end{aligned}$$

Similarly, we can treat the other terms and obtain

$$\begin{aligned} \mathrm{k}_n^{\frac14} L_4^c(q_n \mathcal{B}(\ell_2) q_n, \varphi_n) &= (C_n \otimes_h \mathrm{OH}_n) d_{\psi_n}^{\frac14}, \\ \mathrm{k}_n^{\frac12} L_2(q_n \mathcal{B}(\ell_2) q_n, \varphi_n) &= d_{\psi_n}^{\frac14} (\mathrm{OH}_n \otimes_h \mathrm{OH}_n) d_{\psi_n}^{\frac14}. \end{aligned}$$

In particular, Corollary 2.9 provides a cb-isomorphism

$$\sigma : x \in \mathcal{J}_{\infty,2}(\psi_n) \mapsto \sum_{j=1}^{\mathrm{k}_n} x_j \otimes \delta_j \in L_\infty(\mathcal{A}_n; \mathrm{OH}_{\mathrm{k}_n})$$

onto a cb-complemented subspace with constants independent of n and

$$\big\langle \sigma(x), \omega(y) \big\rangle = \frac{1}{\mathrm{k}_n} \sum_{j=1}^{\mathrm{k}_n} \mathrm{tr}_{\mathcal{A}_n}(x_j^* y_j) = \mathrm{tr}_n(x^* y) = \langle x, y \rangle.$$

In particular, the stated properties of ω follow from those of the mapping σ. $\square$

Now we give a more explicit description of $\mathcal{K}_{1,2}(\psi_n)$. Using the terminology introduced before Lemma 2.11, the dual of the space $\mathcal{L}_\infty^n$ is given by the following direct sum

$$\mathcal{L}_1^n = \big(R_n \otimes_h C_n \big) \oplus_2 \big(R_n \otimes_h \mathrm{OH}_n \big) \oplus_2 \big(\mathrm{OH}_n \otimes_h C_n \big) \oplus_2 \big(\mathrm{OH}_n \otimes_h \mathrm{OH}_n \big).$$

Thus, we may consider the map

$$\Psi_n : \mathcal{L}_1^n \to L_1(q_n \mathcal{B}(\ell_2) q_n)$$

given by

$$\Psi_n(x_1, x_2, x_3, x_4) = x_1 + x_2 d_{\psi_n}^{\frac14} + d_{\psi_n}^{\frac14} x_3 + d_{\psi_n}^{\frac14} x_4 d_{\psi_n}^{\frac14}.$$

Then it is easily checked that $\ker \Psi_n = \mathcal{J}_{\infty,2}(\psi_n)^\perp$ with respect to the anti-linear duality bracket and we deduce $\mathcal{K}_{1,2}(\psi_n) = \mathcal{L}_1^n / \ker \Psi_n$. The finite-dimensional spaces defined so far allow us to take direct limits

$$\mathcal{J}_{\infty,2}(\psi) = \overline{\bigcup_{n \ge 1} \mathcal{J}_{\infty,2}(\psi_n)} \quad \text{and} \quad \mathcal{K}_{1,2}(\psi) = \overline{\bigcup_{n \ge 1} \mathcal{K}_{1,2}(\psi_n)}.$$

Lemma 2.12. *Let $\lambda_1, \lambda_2, \ldots \in \mathbb{R}_+$ be a sequence of strictly positive numbers and define the diagonal operator $\mathsf{d}_\lambda = \sum_k \lambda_k e_{kk}$ on ℓ_2. Let us equip the space graph(d_λ) with the following operator space structures*

$$\begin{aligned} R \cap \ell_2^{oh}(\lambda) &= \mathit{graph}(\mathsf{d}_\lambda) \subset R \oplus_2 \mathrm{OH}, \\ C \cap \ell_2^{oh}(\lambda) &= \mathit{graph}(\mathsf{d}_\lambda) \subset C \oplus_2 \mathrm{OH}. \end{aligned}$$

Then, if we consider the dual spaces

$$\begin{array}{rcl} C+\ell_2^{oh}(\lambda) & = & \big(C \oplus_2 \mathrm{OH}\big) / \big(R \cap \ell_2^{oh}(\lambda)\big)^{\perp}, \\ R+\ell_2^{oh}(\lambda) & = & \big(R \oplus_2 \mathrm{OH}\big) / \big(C \cap \ell_2^{oh}(\lambda)\big)^{\perp}, \end{array}$$

there exists a n.s.s.f. weight ψ on $\mathcal{B}(\ell_2)$ such that

$$\big(R+\ell_2^{oh}(\lambda)\big) \otimes_h \big(C+\ell_2^{oh}(\lambda)\big) = \mathcal{K}_{1,2}(\psi).$$

Proof. If we set

$$q_n = \sum_{k \le n} e_{kk},$$

then we define

$$\begin{array}{rcl} q_n\big(C+\ell_2^{oh}(\lambda)\big) & = & \Big\{ q_n(a,b) + \big(R \cap \ell_2^{oh}(\lambda)\big)^{\perp} \, \big| \, (a,b) \in C \oplus_2 \mathrm{OH} \Big\} \subset C+\ell_2^{oh}(\lambda), \\ q_n\big(R+\ell_2^{oh}(\lambda)\big) & = & \Big\{ q_n(a,b) + \big(C \cap \ell_2^{oh}(\lambda)\big)^{\perp} \, \big| \, (a,b) \in R \oplus_2 \mathrm{OH} \Big\} \subset R+\ell_2^{oh}(\lambda). \end{array}$$

Note that, since the corresponding annihilators are q_n-invariant, these are quotients of $C_n \oplus_2 \mathrm{OH}_n$ and $R_n \oplus_n \mathrm{OH}_n$ respectively. Moreover, recalling that $q_n(x) \to x$ as $n \to \infty$ in the norms of R, OH, C, it is not difficult to see that we may write the Haagerup tensor product $\big(R+\ell_2^{oh}(\lambda)\big) \otimes_h \big(C+\ell_2^{oh}(\lambda)\big)$ as the direct limit

$$\overline{\bigcup_{n \ge 1} q_n(R+\ell_2^{oh}(\lambda)) \otimes_h q_n(C+\ell_2^{oh}(\lambda))}.$$

Therefore, it suffices to show that

$$q_n(R+\ell_2^{oh}(\lambda)) \otimes_h q_n(C+\ell_2^{oh}(\lambda)) = \mathcal{K}_{1,2}(\psi_n),$$

where ψ_n denotes the restriction to $q_n \mathcal{B}(\ell_2) q_n$ of some *n.s.s.f.* weight ψ. However, by duality this is equivalent to see that $q_n(C \cap \ell_2^{oh}(\lambda)) \otimes_h q_n(R \cap \ell_2^{oh}(\lambda)) = \mathcal{J}_{\infty,2}(\psi_n)$ where the spaces $q_n(R \cap \ell_2^{oh}(\lambda)) / q_n(C \cap \ell_2^{oh}(\lambda))$ are the span of

$$\Big\{ (\delta_k, \lambda_k \delta_k) \, \Big| \, 1 \le k \le n \Big\}$$

in $R_n \oplus_2 \mathrm{OH}_n / C_n \oplus_2 \mathrm{OH}_n$ respectively. Indeed, we have

$$\begin{array}{rcl} q_n\big(C+\ell_2^{oh}(\lambda)\big) & = & \big(C_n \oplus_2 \mathrm{OH}_n\big) / q_n(R \cap \ell_2^{oh}(\lambda))^{\perp}, \\ q_n\big(R+\ell_2^{oh}(\lambda)\big) & = & \big(R_n \oplus_2 \mathrm{OH}_n\big) / q_n(C \cap \ell_2^{oh}(\lambda))^{\perp}, \end{array}$$

completely isometrically. Using row/column terminology in terms of matrix units

$$\begin{array}{rcl} q_n\big(C \cap \ell_2^{oh}(\lambda)\big) & = & \mathrm{span}\Big\{ (e_{i1}, \lambda_i e_{i1}) \in C_n \oplus_2 \mathrm{OH}_n \Big\}, \\ q_n\big(R \cap \ell_2^{oh}(\lambda)\big) & = & \mathrm{span}\Big\{ (e_{1j}, \lambda_j e_{1j}) \in R_n \oplus_2 \mathrm{OH}_n \Big\}. \end{array}$$

Therefore, the space $q_n(C \cap \ell_2^{oh}(\lambda)) \otimes_h q_n(R \cap \ell_2^{oh}(\lambda))$ is the subspace

$$\mathrm{span}\Big\{ (e_{ij}, \lambda_j e_{ij}, \lambda_i e_{ij}, \lambda_i \lambda_j e_{ij}) \Big\} = \Big\{ (z, z\mathsf{d}_\lambda, \mathsf{d}_\lambda z, \mathsf{d}_\lambda z \mathsf{d}_\lambda) \, \Big| \, z \in q_n \mathcal{B}(\ell_2) q_n \Big\}$$

of the space $\mathcal{L}_\infty^n$ defined above. Then, we define $\gamma_k \in \mathbb{R}_+$ by the relation $\lambda_k = \gamma_k^{\frac14}$ and consider the *n.s.s.f.* weight ψ on $\mathcal{B}(\ell_2)$ induced by d_γ. In particular, we immediate obtain

$$q_n\big(C \cap \ell_2^{oh}(\lambda)\big) \otimes_h q_n\big(R \cap \ell_2^{oh}(\lambda)\big) = \Big\{ \big(z, z d_{\psi_n}^{\frac14}, d_{\psi_n}^{\frac14} z, d_{\psi_n}^{\frac14} z d_{\psi_n}^{\frac14}\big) \Big\}.$$

The space on the right is by definition $\mathcal{J}_{\infty,2}(\psi_n)$. This completes the proof. $\square$

Proof of Theorem C. By injectivity of the Haagerup tensor product, we may assume that $(\mathrm{X}_1, \mathrm{X}_2) \in \mathcal{Q}(R \oplus_2 \mathrm{OH}) \times \mathcal{Q}(C \oplus_2 \mathrm{OH})$. In particular, the duals X_1^* and X_2^* are subspaces of $C \oplus_2 \mathrm{OH}$ and $R \oplus_2 \mathrm{OH}$ respectively. Therefore, using a well-known result (see Lemma 4.1 below), we may find Hilbert spaces $\mathcal{H}_{ij}$ and $\mathcal{K}_{ij}$ for $i, j = 1, 2$ such that

$$\begin{array}{rcl} \mathrm{X}_1^* & \simeq_{cb} & \mathcal{H}_{11,c} \oplus_2 \mathcal{H}_{12,oh} \oplus_2 graph(\Lambda_1), \\ \mathrm{X}_2^* & \simeq_{cb} & \mathcal{H}_{21,r} \oplus_2 \mathcal{H}_{22,oh} \oplus_2 graph(\Lambda_2), \end{array}$$

where the operators $\Lambda_1 : \mathcal{K}_{11,c} \to \mathcal{K}_{12,oh}$ and $\Lambda_2 : \mathcal{K}_{21,r} \to \mathcal{K}_{22,oh}$ are injective, closed, densely-defined with dense range. On the other hand, using the complete isometries $\mathcal{H}_r^* = \mathcal{H}_c$ and $\mathcal{H}_c^* = \mathcal{H}_r$, we easily obtain the cb-isomorphisms

$$\begin{array}{rcl} \mathrm{X}_1 & \simeq_{cb} & \mathcal{H}_{11,r} \oplus_2 \mathcal{H}_{12,oh} \oplus_2 \Big(\big(\mathcal{K}_{11,r} \oplus_2 \mathcal{K}_{12,oh} \big) / graph(\Lambda_1)^\perp \Big), \\ \mathrm{X}_2 & \simeq_{cb} & \mathcal{H}_{21,c} \oplus_2 \mathcal{H}_{22,oh} \oplus_2 \Big(\big(\mathcal{K}_{21,c} \oplus_2 \mathcal{K}_{22,oh} \big) / graph(\Lambda_2)^\perp \Big). \end{array}$$

Let us set for the sequel

$$\begin{array}{rcl} \mathcal{Z}_1 & = & \big(\mathcal{K}_{11,r} \oplus_2 \mathcal{K}_{12,oh} \big) / graph(\Lambda_1)^\perp, \\ \mathcal{Z}_2 & = & \big(\mathcal{K}_{21,c} \oplus_2 \mathcal{K}_{22,oh} \big) / graph(\Lambda_2)^\perp. \end{array}$$

Then, we have the following cb-isometric inclusion

$$\mathrm{X}_1 \otimes_h \mathrm{X}_2 \subset \mathrm{A}_1 \oplus_2 \mathrm{A}_2 \oplus_2 \mathrm{A}_3 \oplus_2 \mathrm{A}_4 \oplus_2 \mathrm{A}_5 \oplus_2 \mathrm{A}_6, \tag{2.11}$$

where the A_j's are given by

$$\begin{array}{rcl} \mathrm{A}_1 & = & \mathcal{Z}_1 \otimes_h \mathcal{Z}_2 \\ \mathrm{A}_2 & = & \mathcal{H}_{11,r} \otimes_h \mathrm{X}_2 \\ \mathrm{A}_3 & = & \mathrm{X}_1 \otimes_h \mathcal{H}_{21,c} \\ \mathrm{A}_4 & = & \mathcal{H}_{12,oh} \otimes_h \mathcal{Z}_2 \\ \mathrm{A}_5 & = & \mathcal{Z}_1 \otimes_h \mathcal{H}_{22,oh} \\ \mathrm{A}_6 & = & \mathcal{H}_{12,oh} \otimes_h \mathcal{H}_{22,oh}. \end{array}$$

Let us show that the proof can be reduced to the construction of a cb-embedding $\mathcal{Z}_1 \otimes_h \mathcal{Z}_2 \to L_1(\mathcal{A})$ for some QWEP von Neumann algebra $\mathcal{A}$. Indeed, according to [11] we know that OH cb-embeds in $L_1(\mathcal{A})$ for some QWEP type III factor $\mathcal{A}$. Hence, the last term on the right of (2.11) automatically satisfies the assertion. A similar argument works for the second and third terms. Indeed, they clearly embed into $S_1(\mathrm{X}_1)$ and $S_1(\mathrm{X}_2)$ completely isometrically. On the other hand, since $\mathrm{OH} \in \mathcal{QS}(C \oplus R)$ by "Pisier's exercise" and we have by hypothesis

$$\mathrm{X}_1 \in \mathcal{QS}(R \oplus_2 \mathrm{OH}) \quad \text{and} \quad \mathrm{X}_2 \in \mathcal{QS}(C \oplus_2 \mathrm{OH}),$$

both X_1 and X_2 are cb-isomorphic to an element in $\mathcal{QS}(C \oplus R)$. According to [11] one more time, we know that any operator space in $\mathcal{QS}(C \oplus R)$ cb-embeds into $L_1(\mathcal{A})$ for some QWEP von Neumann algebra $\mathcal{A}$. Thus, the spaces $S_1(\mathrm{X}_1)$ and $S_1(\mathrm{X}_2)$ also satisfy the assertion. Finally, for the fourth and fifth terms on the right of (2.11), we may write OH as the graph of a diagonal operator on ℓ_2, see Lemma 4.2 below for further details. In particular, by the self-duality of OH we conclude that these terms can be regarded as particular cases of the first term $\mathcal{Z}_1 \otimes_h \mathcal{Z}_2$. It remains to see that the term $\mathcal{Z}_1 \otimes_h \mathcal{Z}_2$ satisfies the assertion. By discretization (see Lemma 4.5) we may assume that the graphs appearing in the terms $\mathcal{Z}_1$ and

$\mathcal{Z}_2$ above are graphs of diagonal operators d_{λ_1} and d_{λ_2}. Moreover, using polar decomposition we may also assume that both diagonal operators are positive, see the proof of Lemma 4.5 one more time. In fact, by adding a perturbation term we can take the eigenvalues $\lambda_{1k}, \lambda_{2k} \in \mathbb{R}_+$ strictly positive. Indeed, if we replace λ_{jk} by $\xi_{jk} = \lambda_{jk} + \varepsilon_k$ for $j = 1, 2$, the new diagonal operators d_{ξ_1} and d_{ξ_2} satisfy the cb-isomorphisms

$$graph(\mathsf{d}_{\lambda_j}) \simeq_{cb} graph(\mathsf{d}_{\xi_j}) \quad \text{for} \quad j = 1, 2$$

where (arguing as in Lemma 4.2 below) the cb-norms are controlled by

$$\Big(\sum\nolimits_k |\varepsilon_k|^4 \Big)^{\frac{1}{4}}.$$

Therefore, taking the ε_k's small enough, we may write

$$\begin{aligned}
\mathcal{Z}_1 &= \big(R \oplus_2 \mathrm{OH} \big) / \big(C \cap \ell_2^{oh}(\lambda_1) \big)^{\perp} = R + \ell_2^{oh}(\lambda_1) \quad \text{with} \quad \mathsf{d}_{\lambda_1} : C \to \mathrm{OH}, \\
\mathcal{Z}_2 &= \big(C \oplus_2 \mathrm{OH} \big) / \big(R \cap \ell_2^{oh}(\lambda_2) \big)^{\perp} = C + \ell_2^{oh}(\lambda_2) \quad \text{with} \quad \mathsf{d}_{\lambda_2} : R \to \mathrm{OH},
\end{aligned}$$

where the diagonal operators above are positive and invertible. Now we define

$$\lambda_k = \begin{cases} \lambda_{1, \frac{k+1}{2}} & \text{if } k \text{ is odd,} \\ \lambda_{2, \frac{k}{2}} & \text{if } k \text{ is even.} \end{cases}$$

This defines a positive invertible operator d_λ such that

$$\mathcal{Z}_1 \otimes_h \mathcal{Z}_2 \subset \big(R + \ell_2^{oh}(\lambda) \big) \otimes_h \big(C + \ell_2^{oh}(\lambda) \big),$$

where the former is clearly cb-complemented in the latter. According to Lemma 2.12, we conclude that $\mathcal{Z}_1 \otimes_h \mathcal{Z}_2$ can be regarded as a completely complemented subspace of the direct limit

$$\mathcal{K}_{1,2}(\psi) = \overline{\bigcup_{n \ge 1} \mathcal{K}_{1,2}(\psi_n)},$$

for some *n.s.s.f.* weight ψ on $\mathcal{B}(\ell_2)$. It remains to construct a completely isomorphic embedding from $\mathcal{K}_{1,2}(\psi)$ into $L_1(\mathcal{A})$ for some QWEP algebra $\mathcal{A}$. To that aim, letting the constants is such cb-embedding a little perturbation, we may assume without loss of generality that the numbers $\mathrm{k}_n = \psi_n(q_n)$ are non-decreasing positive integers since we may approximate each k_n to its closest integer. This will allow us to apply Lemma 2.11 below. Now, in order to cb-embed $\mathcal{K}_{1,2}(\psi)$ into $L_1(\mathcal{A})$, it suffices to construct a cb-embedding of $\mathcal{K}_{1,2}(\psi_n)$ into $L_1(\mathcal{A}_n')$ for some $\mathcal{A}_n'$ being QWEP and with relevant constants independent of n. Indeed, if so we may consider an ultrafilter $\mathcal{U}$ containing all the intervals (n, ∞), so that we have a completely isometric embedding

$$\mathcal{K}_{1,2}(\psi) = \overline{\bigcup_{n \ge 1} \mathcal{K}_{1,2}(\psi_n)} \to \prod\nolimits_{n, \mathcal{U}} \mathcal{K}_{1,2}(\psi_n).$$

Then, according to [41], our assumption provides a cb-embedding

$$\mathcal{K}_{1,2}(\psi) \to L_1(\mathcal{A}) \quad \text{with} \quad \mathcal{A} = \Big(\prod\nolimits_{n, \mathcal{U}} \mathcal{A}_{n*}' \Big)^*.$$

Moreover, we know from [12] that $\mathcal{A}$ is QWEP provided the $\mathcal{A}_n'$'s are. Therefore, it remains to construct the cb-embeddings $\mathcal{K}_{1,2}(\psi_n) \to L_1(\mathcal{A}_n')$. This follows from

the cb-embedding [11] of OH into $L_1(\mathcal{B})$ for some QWEP type III factor $\mathcal{B}$ and from Lemma 2.11

$$\mathcal{K}_{1,2}(\psi_n) \to L_1(\mathcal{A}_n; \mathrm{OH}_{\mathrm{k}_n}) \to L_1(\mathcal{A}_n \bar{\otimes} \mathcal{B}) = L_p(\mathcal{A}_n').$$

Let us show that $\mathcal{A}_n'$ is QWEP. The algebra $\mathcal{A}_n$ is the free product of k_n copies of $\mathrm{M}_n \oplus \mathrm{M}_n$. Therefore, since we know after [11] and [12] that the QWEP is stable under free products and tensor products, $\mathcal{A}_n' = \mathcal{A}_n \otimes \mathcal{B}$ is QWEP. □

Corollary 2.13. *S_q cb-embeds into $L_1(\mathcal{A})$ for some* QWEP *algebra $\mathcal{A}$.*

Proof. Using the complete isometry

$$S_q = C_q \otimes_h R_q,$$

the assertion follows combining Lemma 1.1 and Theorem C. □

3. Mixed norms of free variables

In this section we present a variation of the free Rosenthal inequality [18]. This is the main result of [16] and will be a key point to prove the complete embedding of L_q into L_p in the general case. Its statement forces us to introduce some new classes of noncommutative function spaces. The motivation comes from our construction in the previous section and some classical probabilistic estimates.

3.1. **Conditional L_p spaces.** Inspired by Pisier's theory [33] of noncommutative vector-valued L_p spaces, several noncommutative function spaces have been recently introduced in quantum probability. The first insight came from some of Pisier's fundamental equalities, which we briefly review. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two hyperfinite von Neumann algebras. Given $1 \le p, q \le \infty$, we define $1/r = |1/p - 1/q|$. If $p \le q$, the norm of x in $L_p(\mathcal{N}_1; L_q(\mathcal{N}_2))$ is given by

$$\inf \Big\{ \|\alpha\|_{L_{2r}(\mathcal{N}_1)} \|y\|_{L_q(\mathcal{N}_1 \bar{\otimes} \mathcal{N}_2)} \|\beta\|_{L_{2r}(\mathcal{N}_1)} \ \big| \ x = \alpha y \beta \Big\}. \tag{3.1}$$

If $p \ge q$, the norm of $x \in L_p(\mathcal{N}_1; L_q(\mathcal{N}_2))$ is given by

$$\sup \Big\{ \|\alpha x \beta\|_{L_q(\mathcal{N}_1 \bar{\otimes} \mathcal{N}_2)} \ \big| \ \alpha, \beta \in \mathsf{B}_{L_{2r}(\mathcal{N}_1)} \Big\}. \tag{3.2}$$

The hyperfiniteness is an essential assumption in [33]. However, when dealing with mixed $L_p(L_q)$ norms, Pisier's identities remain true for general von Neumann algebras, see [21]. On the other hand, the *row* and *column* subspaces of L_p are defined as follows

$$R_p^n(L_p(\mathcal{M})) = \Big\{ \sum_{k=1}^n x_k \otimes e_{1k} \ \big| \ x_k \in L_p(\mathcal{M}) \Big\} \subset L_p\big(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2)\big),$$

$$C_p^n(L_p(\mathcal{M})) = \Big\{ \sum_{k=1}^n x_k \otimes e_{k1} \ \big| \ x_k \in L_p(\mathcal{M}) \Big\} \subset L_p\big(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2)\big),$$

where (e_{ij}) denotes the unit vector basis of $\mathcal{B}(\ell_2)$. These spaces are crucial in the noncommutative Khintchine/Rosenthal type inequalities [18, 24, 27] and in noncommutative martingale inequalities [19, 28, 38], where the row and column

spaces are traditionally denoted by $L_p(\mathcal{M};\ell_2^r)$ and $L_p(\mathcal{M};\ell_2^c)$. The norm in these spaces is given by

$$\Big\|\sum_{k=1}^n x_k\otimes e_{1k}\Big\|_{R_p^n(L_p(\mathcal{M}))} = \Big\|\Big(\sum_{k=1}^n x_kx_k^*\Big)^{\frac12}\Big\|_{L_p(\mathcal{M})},$$
$$\Big\|\sum_{k=1}^n x_k\otimes e_{k1}\Big\|_{C_p^n(L_p(\mathcal{M}))} = \Big\|\Big(\sum_{k=1}^n x_k^*x_k\Big)^{\frac12}\Big\|_{L_p(\mathcal{M})}.$$

Remark 3.1. In what follows we shall write

$$R_p^n(L_p(\mathcal{M})) = L_p(\mathcal{M};R_p^n) \quad \text{and} \quad C_p^n(L_p(\mathcal{M})) = L_p(\mathcal{M};C_p^n).$$

Now, let us assume that $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ and that there exists a *n.f.* conditional expectation $\mathsf{E}:\mathcal{M}\to\mathcal{N}$. Then we may define L_p norms of the *conditional square functions*

$$\Big(\sum_{k=1}^n \mathsf{E}(x_kx_k^*)\Big)^{\frac12} \quad \text{and} \quad \Big(\sum_{k=1}^n \mathsf{E}(x_k^*x_k)\Big)^{\frac12}.$$

The expressions $\mathsf{E}(x_kx_k^*)$ and $\mathsf{E}(x_k^*x_k)$ have to be defined properly for $1\le p\le 2$, see [10] or Chapter 1 of [16]. Note that the resulting spaces coincide with the row and column spaces defined above when $\mathcal{N}$ is $\mathcal{M}$ itself. When $n=1$ we recover the spaces $L_p^r(\mathcal{M},\mathsf{E})$ and $L_p^c(\mathcal{M},\mathsf{E})$, which have been instrumental in proving Doob's inequality [10], see also [20] for more applications. In particular, taking $\mathcal{M}_{\oplus n}$ to be the n-fold direct sum $\mathcal{M}\oplus\mathcal{M}\oplus\cdots\oplus\mathcal{M}$ and considering the conditional expectation

$$\mathcal{E}_n : \sum_{k=1}^n x_k\otimes\delta_k\in\mathcal{M}_{\oplus n}\mapsto \frac1n\sum_{k=1}^n x_k\in\mathcal{M},$$

we easily obtain the following isometric isomorphisms

$$\begin{aligned} L_p(\mathcal{M};R_p^n) &= \sqrt{n}\,L_p^r\big(\mathcal{M}_{\oplus n},\mathcal{E}_n\big),\\ L_p(\mathcal{M};C_p^n) &= \sqrt{n}\,L_p^c\big(\mathcal{M}_{\oplus n},\mathcal{E}_n\big). \end{aligned} \tag{3.3}$$

We have already introduced $L_p(L_q)$ spaces, row and column subspaces of L_p and some variations associated to a given conditional expectation. The careful reader may have noticed that many of these norms have been used in Section 2. Now we present a unified approach for this kind of spaces. All the norms considered so far fit into more general families of noncommutative function spaces, which we now define. Let us consider the solid K in $\mathbb{R}^3$ defined by

$$\mathsf{K} = \Big\{(1/u,1/v,1/q)\,\big|\, 2\le u,v\le\infty,\ 1\le q\le\infty,\ 1/u+1/q+1/v\le 1\Big\}.$$

Let $\mathcal{M}$ be a von Neumann algebra equipped with a *n.f.* state φ and let $\mathcal{N}$ be a given von Neumann subalgebra with corresponding conditional expectation E. The amalgamated and conditional L_p spaces are defined as follows.

(i) Let $(1/u,1/v,1/q)\in\mathsf{K}$ and take $1/p=1/u+1/q+1/v$. Then we define the *amalgamated L_p space* associated to the indices (u,q,v) as the subspace $L_u(\mathcal{N})L_q(\mathcal{M})L_v(\mathcal{N})$ of $L_p(\mathcal{M})$. The norm is given by

$$\inf\Big\{\|\alpha\|_{L_u(\mathcal{N})}\|y\|_{L_q(\mathcal{M})}\|\beta\|_{L_v(\mathcal{N})}\,\big|\, x=\alpha y\beta\Big\}.$$

(ii) Let $(1/u, 1/v, 1/p) \in \mathsf{K}$ and take $1/s = 1/u + 1/p + 1/v$. Then we define the *conditional* L_p *space* associated to the indices (u,v) as the completion of $L_p(\mathcal{M})$ with respect to the following norm

$$\sup \Big\{ \|axb\|_{L_s(\mathcal{M})} \, \big| \, \|a\|_{L_u(\mathcal{N})}, \|b\|_{L_v(\mathcal{N})} \leq 1 \Big\}.$$

This space will be denoted by

$$L^p_{(u,v)}(\mathcal{M}, \mathsf{E}).$$

The reader is referred to [16] for a much more detailed exposition of these spaces. In the following, it will also be useful to recognize some important spaces in the terminology just introduced. Here are the basic examples of amalgamated and conditional L_p spaces. The non-trivial isometric identities below, which will be used in the following with no further reference, are proved in [16].

(a) The spaces $L_p(\mathcal{M})$ satisfy

$$L_p(\mathcal{M}) = L_\infty(\mathcal{N}) L_p(\mathcal{M}) L_\infty(\mathcal{N}) \quad \text{and} \quad L_p(\mathcal{M}) = L^p_{(\infty,\infty)}(\mathcal{M}, \mathsf{E}).$$

(b) The spaces $L_p\big(\mathcal{N}_1; L_q(\mathcal{N}_2)\big)$:

- If $p \leq q$ and $1/r = 1/p - 1/q$, we have
$$L_p\big(\mathcal{N}_1; L_q(\mathcal{N}_2)\big) = L_{2r}(\mathcal{N}_1) L_q(\mathcal{N}_1 \bar{\otimes} \mathcal{N}_2) L_{2r}(\mathcal{N}_1).$$
- If $p \geq q$ and $1/r = 1/q - 1/p$, we have
$$L_p\big(\mathcal{N}_1; L_q(\mathcal{N}_2)\big) = L^p_{(2r,2r)}(\mathcal{N}_1 \bar{\otimes} \mathcal{N}_2, \mathsf{E}),$$
where $\mathsf{E} : \mathcal{N}_1 \bar{\otimes} \mathcal{N}_2 \to \mathcal{N}_1$ is given by $\mathsf{E} = id_{\mathcal{N}_1} \otimes \varphi_{\mathcal{N}_2}$.

(c) The spaces $L_p^r(\mathcal{M}, \mathsf{E})$ and $L_p^c(\mathcal{M}, \mathsf{E})$:

- If $1 \leq p \leq 2$ and $1/p = 1/2 + 1/s$, we have
$$\begin{aligned} L_p^r(\mathcal{M}, \mathsf{E}) &= L_s(\mathcal{N}) L_2(\mathcal{M}) L_\infty(\mathcal{N}), \\ L_p^c(\mathcal{M}, \mathsf{E}) &= L_\infty(\mathcal{N}) L_2(\mathcal{M}) L_s(\mathcal{N}). \end{aligned}$$
- If $2 \leq p \leq \infty$ and $1/p + 1/s = 1/2$, we have
$$\begin{aligned} L_p^r(\mathcal{M}, \mathsf{E}) &= L^p_{(s,\infty)}(\mathcal{M}, \mathsf{E}), \\ L_p^c(\mathcal{M}, \mathsf{E}) &= L^p_{(\infty,s)}(\mathcal{M}, \mathsf{E}). \end{aligned}$$

By (3.3), we have also representations for $L_p(\mathcal{M}, R_p^n)$ and $L_p(\mathcal{M}, C_p^n)$.

(d) Along the paper, we shall also find representations of asymmetric spaces $L_{(u,v)}(\mathcal{M})$ (a non-standard operator space structure on L_p defined below which will be crucial in this paper) in terms of either amalgamated or conditional L_p spaces. This will be a key point since we need to handle spaces of the form $S_p(L_{(u,v)}(\mathcal{M}))$. The use of amalgamated L_p spaces or conditional L_p spaces depends on the sign of the term $1/u + 1/v - 1/p$.

Now we collect the complex interpolation and duality properties of amalgamated and conditional L_p spaces from [16]. Our interpolation identities generalize some previous results by Pisier [31] and very recently by Xu [53]. We need to consider the following subset of the solid K

$$\mathsf{K}_0 = \Big\{ (1/u, 1/v, 1/q) \in \mathsf{K} \, \big| \, 2 < u, v \leq \infty, \ 1 < q < \infty, \ 1/u + 1/q + 1/v < 1 \Big\}.$$

Theorem 3.2. *The following properties hold:*

- **a)** *If* $(1/u, 1/v, 1/q) \in \mathsf{K}$, $L_u(\mathcal{N})L_q(\mathcal{M})L_v(\mathcal{N})$ *is a Banach space.*
- **b)** *If* $(1/u_j, 1/v_j, 1/q_j) \in \mathsf{K}$ *for* $j = 0, 1$ *and*

$$(1/u_\theta, 1/v_\theta, 1/q_\theta) = \sum_{j=0,1} |1-j-\theta| (1/u_j, 1/v_j, 1/q_j),$$

the space $L_{u_\theta}(\mathcal{N})L_{q_\theta}(\mathcal{M})L_{v_\theta}(\mathcal{N})$ *is isometrically isomorphic to*

$$\Big[L_{u_0}(\mathcal{N})L_{q_0}(\mathcal{M})L_{v_0}(\mathcal{N}), L_{u_1}(\mathcal{N})L_{q_1}(\mathcal{M})L_{v_1}(\mathcal{N}) \Big]_\theta .$$

- **c)** *If* $(1/u, 1/v, 1/q) \in \mathsf{K}_0$ *and* $1 - 1/p = 1/u + 1/q + 1/v$*, we have*

$$\big(L_u(\mathcal{N})L_q(\mathcal{M})L_v(\mathcal{N}) \big)^* = L_{(u,v)}^p(\mathcal{M}, \mathsf{E}),$$

$$\big(L_{(u,v)}^p(\mathcal{M}, \mathsf{E}) \big)^* = L_u(\mathcal{N})L_q(\mathcal{M})L_v(\mathcal{N}).$$

- **d)** *In particular, we obtain the following isometric isomorphisms*

$$\Big[L_{(u_0,v_0)}^{p_0}(\mathcal{M}, \mathsf{E}), L_{(u_1,v_1)}^{p_1}(\mathcal{M}, \mathsf{E}) \Big]_\theta = L_{(u_\theta,v_\theta)}^{p_\theta}(\mathcal{M}, \mathsf{E}).$$

In the following result we list some particular cases of Theorem 3.2 under the restriction $p_0 = p_1$, since these are the main interpolation identities used in this paper. The case where both p_0 and p_1 are ∞ is excluded in Theorem 3.2. That case was stated in Lemma 2.5 above and the proof was also given in [16].

Corollary 3.3. *If* $2 \le p < \infty$*, we set*

$$(1/u, 1/v) = (\theta/q, (1-\theta)/q) \quad \text{for} \quad 0 < \theta < 1 \text{ and } q \text{ given by} \quad 1/2 = 1/p + 1/q.$$

Then, the following isometric isomorphisms hold

$$\begin{array}{rcl} \big[L_p(\mathcal{M}), L_p^r(\mathcal{M}, \mathsf{E}) \big]_\theta & = & L_{(u,\infty)}^p(\mathcal{M}, \mathsf{E}), \\ \big[L_p^c(\mathcal{M}, \mathsf{E}), L_p(\mathcal{M}) \big]_\theta & = & L_{(\infty,v)}^p(\mathcal{M}, \mathsf{E}), \\ \big[L_p^c(\mathcal{M}, \mathsf{E}), L_p^r(\mathcal{M}, \mathsf{E}) \big]_\theta & = & L_{(u,v)}^p \, (\mathcal{M}, \mathsf{E}). \end{array}$$

3.2. **A variant of free Rosenthal's inequality.** In this paragraph we study a variation of the free Rosenthal inequality [18], which will be applied in the sequel. Intersection of L_p spaces appear naturally in the theory of noncommutative Hardy spaces. These spaces are also natural byproducts of Rosenthal's inequality for sums of independent random variables. Let us first illustrate this point in the commutative setting and then provide the link to the spaces defined above. Let $g_1, g_2, \ldots, g_n$ be a finite collection of independent random variables on a probability space (Ω, μ). If $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ is an independent family of Bernoullis equidistributed on ± 1, the Khintchine inequality implies for $0 < s < \infty$ that

$$\Big(\int_\Omega \Big[\sum_{k=1}^n |g_k|^2 \Big]^{\frac{s}{2}} d\mu \Big)^{\frac{1}{s}} \sim_{c_s} \mathbb{E} \Big\| \sum_{k=1}^n \varepsilon_k g_k \Big\|_s .$$

Therefore, Rosenthal's inequality [42] gives for $2 \le s < \infty$

$$\Big(\int_\Omega \Big[\sum_{k=1}^n |g_k|^2 \Big]^{\frac{s}{2}} d\mu \Big)^{\frac{1}{s}} \sim_{c_s} \Big(\sum_{k=1}^n \|g_k\|_s^s \Big)^{\frac{1}{s}} + \Big(\sum_{k=1}^n \|g_k\|_2^2 \Big)^{\frac{1}{2}} . \tag{3.4}$$

Now, given $1 \le q \le p < \infty$ and an independent family $f_1, f_2, \dots f_n$ of p-integrable random variables, we define $g_k = |f_k|^{q/2}$ for $1 \le k \le n$. Then we have the following identity for the index $s = 2p/q$

$$\Big(\int_\Omega \Big[\sum_{k=1}^n |f_k|^q \Big]^{\frac{p}{q}} d\mu \Big)^{\frac{1}{p}} = \Big(\int_\Omega \Big[\sum_{k=1}^n |g_k|^2 \Big]^{\frac{s}{2}} d\mu \Big)^{\frac{2}{qs}}. \tag{3.5}$$

Since the g_k's are independent and $2 \le s < \infty$, (3.4) and (3.5) give

$$\Big(\int_\Omega \Big(\sum_{k=1}^n |f_k|^q \Big)^{\frac{p}{q}} d\mu \Big)^{\frac{1}{p}} \sim_{c_p} \Big(\sum_{k=1}^n \|f_k\|_p^p \Big)^{\frac{1}{p}} + \Big(\sum_{k=1}^n \|f_k\|_q^q \Big)^{\frac{1}{q}}. \tag{Σ_{pq}}$$

In particular, (Σ_{pq}) provides a natural way to realize the space

$$\mathcal{J}_{p,q}^n(\Omega) = n^{\frac{1}{p}} L_p(\Omega) \cap n^{\frac{1}{q}} L_q(\Omega)$$

as an isomorph of a subspace of $L_p(\Omega; \ell_q^n)$. More precisely, if $f_1, f_2, \dots, f_n$ are taken to be independent copies of a given random variable f, the right hand side of (Σ_{pq}) is the norm of f in the intersection space $\mathcal{J}_{p,q}^n(\Omega)$ and inequality (Σ_{pq}) provides an isomorphic embedding

$$f \in \mathcal{J}_{p,q}^n(\Omega) \mapsto (f_1, f_2, \dots, f_n) \in L_p(\Omega; \ell_q^n).$$

Quite surprisingly, replacing independent variables by matrices of independent variables in (Σ_{pq}) requires to intersect *four* spaces using the so-called *asymmetric* L_p spaces. In other words, the natural operator space structure of $\mathcal{J}_{p,q}^n$ comes from a 4-term intersection space. This phenomenon was discovered for the first time in [15] and we have already met it in Section 2. To justify this point, instead of giving precise definitions we note that Hölder inequality gives $L_p = L_{2p} L_{2p}$, meaning that the p-norm of f is the infimum of $\|g\|_{2p} \|h\|_{2p}$ over all possible factorizations $f = gh$. If L_{2p}^r and L_{2p}^c denote the row and column quantizations (2.8) of L_{2p}, the operator space analogue of this isometry is given by the complete isometry

$$L_p = L_{2p}^r L_{2p}^c.$$

This will be further explained below. In particular, according to the algebraic definition of $L_p(\ell_q)$, the intersection space $\mathcal{J}_{p,q}^n$ has to be redefined as the product

$$\mathcal{J}_{p,q}^n = \Big(n^{\frac{1}{2p}} L_{2p}^r \cap n^{\frac{1}{2q}} L_{2q}^r \Big) \Big(n^{\frac{1}{2p}} L_{2p}^c \cap n^{\frac{1}{2q}} L_{2q}^c \Big).$$

According to [16], we find

$$\mathcal{J}_{p,q}^n = n^{\frac{1}{p}} L_{2p}^r L_{2p}^c \cap n^{\frac{1}{2p} + \frac{1}{2q}} L_{2p}^r L_{2q}^c \cap n^{\frac{1}{2q} + \frac{1}{2p}} L_{2q}^r L_{2p}^c \cap n^{\frac{1}{q}} L_{2q}^r L_{2q}^c. \tag{3.6}$$

Of course, these notions are not rigorously defined and will be analyzed in more detail below. Our only aim here is to motivate the forthcoming results. Let us now see how the space in (3.6) generalizes our first definition of $\mathcal{J}_{p,q}^n(\Omega)$. On the Banach space level we have the isometries

$$L_{2p}^r L_{2q}^c = L_s = L_{2q}^r L_{2p}^c \quad \text{with} \quad 1/s = 1/2p + 1/2q.$$

Moreover, again by Hölder inequality it is clear that

$$n^{\frac{1}{s}} \|f\|_s \le \max \Big\{ n^{\frac{1}{p}} \|f\|_p, n^{\frac{1}{q}} \|f\|_q \Big\}.$$

Therefore, the two cross terms in the middle of (3.6) disappear in the Banach space level. However, as we shall see in this section, in the category of operator spaces

all the four terms have a significant contribution. The operator space/free version of (Σ_{pq}) is the main result in [16]. It is worthy of mention that this result goes a bit further than its commutative counterpart. More precisely, in contrast with the classical case, we find a *right* formulation for $(\Sigma_{\infty q})$. Indeed, as it happens with the Khintchine and Rosenthal inequalities, the limit case as $p \to \infty$ holds when replacing independence by Voiculescu's concept of freeness [49]. Unfortunately, the techniques for $(p,q) = (\infty, 2)$ used in Section 2 do not apply in the general case and the arguments in [16] become more involved. This is mainly because a concrete Fock space representation does not seem available for $L_p(\mathcal{A})$ with $\mathcal{A}$ a free product algebra and $p < \infty$. Therefore, a *purely free* proof of (Σ_{pq}) seems out of the scope by now. Nevertheless, since we shall need to be familiar with the results in [16], we summarize them here. We observe in advance that all the spaces and results presented in this paragraph for $1 \le q \le p \le \infty$ are consistent with their corresponding versions for $(p,q) = (\infty, 2)$ used in Section 2.

Now, if $2 \le u, v \le \infty$ and $1/p = 1/u + 1/v$ for some $1 \le p \le \infty$, we define the *asymmetric L_p space* associated to the pair (u,v) as the $\mathcal{M}$-amalgamated Haagerup tensor product

$$L_{(u,v)}(\mathcal{M}) = L_u^r(\mathcal{M}) \otimes_{\mathcal{M},h} L_v^c(\mathcal{M}). \tag{3.7}$$

That is, we consider the quotient of $L_u^r(\mathcal{M}) \otimes_h L_v^c(\mathcal{M})$ by the closed subspace $\mathcal{I}$ generated by the differences $x_1\gamma \otimes x_2 - x_1 \otimes \gamma x_2$ with $\gamma \in \mathcal{M}$. Recall that the row and column operator space structures on $L_u(\mathcal{M})$ and $L_v(\mathcal{M})$ have been already defined in (2.8). By a well-known factorization argument, see e.g. Lemma 3.5 in [33], the norm of an element x in $L_{(u,v)}(\mathcal{M})$ is given by

$$\|x\|_{(u,v)} = \inf_{x=\alpha\beta} \|\alpha\|_{L_u(\mathcal{M})} \|\beta\|_{L_v(\mathcal{M})}.$$

Remark 3.4. We have a complete isometry $L_p(\mathcal{M}) = L_{(2p,2p)}(\mathcal{M})$.

Remark 3.5. Asymmetric L_p spaces were introduced in [15] for matrix algebras. In fact, if $\mathcal{M}$ is the algebra M_m of $m \times m$ matrices, we define the *asymmetric Schatten p-class* as follows

$$S_{(u,v)}^m = C_{u/2}^m \otimes_h R_{v/2}^m.$$

As observed in [16], this definition is consistent with our definition (3.7).

According to the discussion which led to (3.6), we know how the general aspect of $\mathcal{J}_{p,q}^n(\mathcal{M})$ should be. Now, equipped with asymmetric L_p spaces we know how to factorize noncommutative L_p spaces in the right way and define

$$\mathcal{J}_{p,q}^n(\mathcal{M}) = \bigcap_{u,v \in \{2p,2q\}} n^{\frac{1}{2p}+\frac{1}{2q}} L_{(u,v)}(\mathcal{M}).$$

The following result generalizes (2.10), see [16] for the proof.

Lemma 3.6. *If we take*

$$\mathcal{M}_m = \mathrm{M}_m(\mathcal{M}) \quad \textit{and} \quad \mathsf{E}_m = id_{\mathrm{M}_m} \otimes \varphi : \mathcal{M}_m \to \mathrm{M}_m$$

for $m \ge 1$ and consider the index $1/r = 1/q - 1/p$, we have an isometry

$$S_p^m\big(\mathcal{J}_{p,q}^n(\mathcal{M})\big) = \bigcap_{u,v \in \{2r,\infty\}} n^{\frac{1}{u}+\frac{1}{p}+\frac{1}{v}} L_{(u,v)}^p(\mathcal{M}_m, \mathsf{E}_m).$$

Remark 3.7. According to Lemma 3.6, we set

$$\mathcal{J}_{p,q}^n(\mathcal{M},\mathsf{E}) = \bigcap_{u,v \in \{2r,\infty\}} n^{\frac{1}{u}+\frac{1}{p}+\frac{1}{v}} L_{(u,v)}^p(\mathcal{M},\mathsf{E}).$$

Lemma 3.6 shows us the way to work in what follows. Indeed, instead of working with the o.s.s. of the spaces $\mathcal{J}_{p,q}^n(\mathcal{M})$, it suffices to argue with the Banach space structure of the more general spaces $\mathcal{J}_{p,q}^n(\mathcal{M},\mathsf{E})$. In this spirit, for $1 \le q \le p \le \infty$ we set $1/r = 1/q - 1/p$ and introduce the spaces

$$\begin{array}{rcl} \mathcal{R}_{2p,q}^n(\mathcal{M},\mathsf{E}) & = & n^{\frac{1}{2p}} L_{2p}(\mathcal{M}) \cap n^{\frac{1}{2q}} L_{(2r,\infty)}^{2p}(\mathcal{M},\mathsf{E}), \\ \mathcal{C}_{2p,q}^n(\mathcal{M},\mathsf{E}) & = & n^{\frac{1}{2p}} L_{2p}(\mathcal{M}) \cap n^{\frac{1}{2q}} L_{(\infty,2r)}^{2p}(\mathcal{M},\mathsf{E}). \end{array}$$

Remark 3.8. Let $(\mathrm{X}_1,\mathrm{X}_2)$ be a pair of operator spaces containing a von Neumann algebra $\mathcal{M}$ as a common two-sided ideal. We define the *amalgamated* Haagerup tensor product $\mathrm{X}_1 \otimes_{\mathcal{M},h} \mathrm{X}_2$ as the quotient of $\mathrm{X}_1 \otimes_h \mathrm{X}_2$ by the closed subspace $\mathcal{I}$ generated by the differences $x_1\gamma \otimes x_2 - x_1 \otimes \gamma x_2$ with $\gamma \in \mathcal{M}$. This notion has already been used above in the definition of asymmetric L_p spaces. Let us write $\mathrm{X}_1 \otimes_{\mathcal{M}} \mathrm{X}_2$ to denote the underlying Banach space of $\mathrm{X}_1 \otimes_{\mathcal{M},h} \mathrm{X}_2$. Our definition uses the operator space structure of the X_j's since the row (resp. column) square functions are not necessarily closed operations in X_1 (resp. X_2). However, in the sequel it will be important to note that much less structure on $(\mathrm{X}_1,\mathrm{X}_2)$ is needed to define the norm in $\mathrm{X}_1 \otimes_{\mathcal{M}} \mathrm{X}_2$. Indeed, we just need to impose conditions under which the row and column square functions become closed operations in X_1 and X_2 respectively. In particular, this is guaranteed if X_1 is a right $\mathcal{M}$-module and X_2 is a left $\mathcal{M}$-module, see Chapter 6 of [16] for further details. Therefore, we may define the Banach space

$$\mathcal{R}_{2p,q}^n(\mathcal{M},\mathsf{E}) \otimes_{\mathcal{M}} \mathcal{C}_{2p,q}^n(\mathcal{M},\mathsf{E}).$$

The theorem below collects the key results in [16].

Theorem 3.9. *The following isomorphisms hold*:

a) *If $1 \le p \le \infty$ and $1/q = 1-\theta+\theta/p$, we have*

$$\begin{array}{rcl} \big[\mathcal{R}_{2p,1}^n(\mathcal{M},\mathsf{E}), \mathcal{R}_{2p,p}^n(\mathcal{M},\mathsf{E})\big]_\theta & \simeq & \mathcal{R}_{2p,q}^n(\mathcal{M},\mathsf{E}), \\ \big[\mathcal{C}_{2p,1}^n(\mathcal{M},\mathsf{E}), \mathcal{C}_{2p,p}^n(\mathcal{M},\mathsf{E})\big]_\theta & \simeq & \mathcal{C}_{2p,q}^n(\mathcal{M},\mathsf{E}). \end{array}$$

b) *If $1 \le p \le \infty$, we have*

$$\big[\mathcal{R}_{2p,1}^n(\mathcal{M},\mathsf{E}), \mathcal{C}_{2p,1}^n(\mathcal{M},\mathsf{E})\big]_\theta \simeq \bigcap_{u,v \in \{2p',\infty\}} n^{\frac{1-\theta}{u}+\frac{1}{2p}+\frac{\theta}{v}} L^{2p}_{(\frac{u}{1-\theta},\frac{v}{\theta})}(\mathcal{M},\mathsf{E}).$$

c) *If $1 \le q \le p \le \infty$, we have*

$$\mathcal{J}_{p,q}^n(\mathcal{M},\mathsf{E}) \simeq \mathcal{R}_{2p,q}^n(\mathcal{M},\mathsf{E}) \otimes_{\mathcal{M}} \mathcal{C}_{2p,q}^n(\mathcal{M},\mathsf{E}).$$

d) *If $1 \le p \le \infty$ and $1/q = 1-\theta+\theta/p$, we have*

$$\mathcal{J}_{p,q}^n(\mathcal{M},\mathsf{E}) \simeq \big[\mathcal{J}_{p,1}^n(\mathcal{M},\mathsf{E}), \mathcal{J}_{p,p}^n(\mathcal{M},\mathsf{E})\big]_\theta.$$

Moreover, the involved relevant constants are in all cases independent of n.

Of course, the main result in the proof of the free analogue of (Σ_{pq}) is the last interpolation isomorphism in (d). In contrast with the case $(p,q) = (\infty, 2)$, its proof does not follow from (b) but from the combination of (a) and (c). Indeed, the right hand side of (b) only gives a $\mathcal{J}$-space when $\theta = 1/2$. This is related to Remark 2.10 above. The following corollary gives the free analogue of (Σ_{pq}) in the operator space case ($\mathcal{J}_{p,q}^n(\mathcal{M})$ spaces) and in the amalgamated case ($\mathcal{J}_{p,q}^n(\mathcal{M}, \mathsf{E})$ spaces). To that aim we take again $\mathsf{A}_k = \mathcal{M} \oplus \mathcal{M}$ for $1 \le k \le n$.

Corollary 3.10. *If $\mathcal{A}_\mathcal{N} = *_\mathcal{N} \mathsf{A}_k$, the map*

$$u : x \in \mathcal{J}_{p,q}^n(\mathcal{M}, \mathsf{E}) \mapsto \sum_{k=1}^n x_k \otimes \delta_k \in L_p(\mathcal{A}_\mathcal{N}; \ell_q^n)$$

is an isomorphism with complemented image and constants independent of n. In particular, replacing as usual $(\mathcal{M}, \mathcal{N}, \mathsf{E})$ by $(\mathcal{M}_m, \mathrm{M}_m, \mathsf{E}_m)$ and replacing $\mathcal{A}_\mathcal{N}$ by the non-amalgamated algebra $\mathcal{A}_\mathbb{C} = \mathsf{A}_1 * \mathsf{A}_2 * \cdots * \mathsf{A}_n$, we obtain a cb-isomorphism with cb-complemented image and constants independent of n

$$(\Sigma_{pq}) \qquad \sigma : x \in \mathcal{J}_{p,q}^n(\mathcal{M}) \mapsto \sum_{k=1}^n x_k \otimes \delta_k \in L_p(\mathcal{A}_\mathbb{C}; \ell_q^n).$$

4. Construction of the main embedding

Now we have all the tools to prove our main result. In the first paragraph we embed the Schatten class S_q into $L_p(\mathcal{A})$ for some QWEP von Neumann algebra $\mathcal{A}$. Roughly speaking, the proof is almost identical to our original argument after replacing Corollary 2.9 by Corollary 3.10. In particular, we shall omit some details in our construction and include some others, such as the proofs of Lemmas 4.2 and 4.5 which we applied in our proof of Theorem C. The second paragraph is devoted to the stability of hyperfiniteness and there we will present the transference argument mentioned in the Introduction. Finally, the last paragraph contains our construction for general von Neumann algebras.

4.1. **Embedding Schatten classes.** We being by embedding the Schatten class S_q into $L_p(\mathcal{A})$ for some QWEP von Neumann algebra. In fact, we shall prove a more general statement (a generalization of Theorem C) for which we need some preliminaries. Although the following results might be well-known, we state them in detail since they will be key tools in our construction. The next lemma has been known to Xu and the first-named author for quite some time. We refer to Xu's paper [51] for an even more general statement than the result presented below.

Lemma 4.1. *Given $1 \le p \le \infty$ and a closed subspace X of $R_p \oplus_2 \mathrm{OH}$, there exist closed subspaces $\mathcal{H}_1, \mathcal{H}_2, \mathcal{K}_1, \mathcal{K}_2$ of ℓ_2 and an injective closed densely-defined operator $\Lambda : \mathcal{K}_1 \to \mathcal{K}_2$ with dense range such that*

$$\mathrm{X} \simeq_{cb} \mathcal{H}_{1,r_p} \oplus_2 \mathcal{H}_{2,oh} \oplus_2 graph(\Lambda),$$

where the graph of Λ is regarded as a subspace of $\mathcal{K}_{1,r_p} \oplus_2 \mathcal{K}_{2,oh}$ and the relevant constants in the complete isomorphism above do not depend on the subspace X. Moreover, since $R_p = C_{p'}$ the same result can be written in terms of column spaces.

In the following, we shall also need to recognize Pisier's operator Hilbert space OH as the graph of certain diagonal operator on ℓ_2. More precisely, the following result will be used below.

Lemma 4.2. *Given $1 \le p \le \infty$, there exists a sequence $\lambda_1, \lambda_2, \ldots$ in $\mathbb{R}_+$ for which the associated diagonal map $\mathsf{d}_\lambda = \sum_k \lambda_k e_{kk} : R_p \to \mathrm{OH}$ satisfies the following complete isomorphism*

$$\mathrm{OH} \simeq_{cb} \mathit{graph}(\mathsf{d}_\lambda).$$

Proof. Let us define

$$u : \delta_k \in \mathrm{OH} \mapsto (\lambda_k^{-1}\delta_k, \delta_k) \in \mathit{graph}(\mathsf{d}_\lambda).$$

The mapping u establishes a linear isomorphism between OH and $\mathit{graph}(\mathsf{d}_\lambda)$. The inverse map of u is the coordinate projection into the second component, which is clearly a complete contraction. Regarding the cb-norm of u, since $\mathit{graph}(\mathsf{d}_\lambda)$ is equipped with the o.s.s. of $R_p \oplus_2 \mathrm{OH}$, we have

$$\|u\|_{cb} = \sqrt{1+\xi^2}$$

with ξ standing for the cb-norm of $\mathsf{d}_{\lambda^{-1}} : \mathrm{OH} \to R_p$. We claim that

$$\xi \le \Big(\sum\nolimits_k |\lambda_k^{-1}|^4 \Big)^{\frac14},$$

so that it suffices to take $\lambda_1, \lambda_2, \ldots$ large enough to deduce the assertion. Indeed, it is well-known that the inequality above holds for the map $\mathsf{d}_{\lambda^{-1}} : \mathrm{OH} \to R$ and also for $\mathsf{d}_{\lambda^{-1}} : \mathrm{OH} \to C$. Therefore, our claim follows by complex interpolation. □

Remark 4.3. The constants in Lemma 4.2 are uniformly bounded on p.

The main embedding result in Xu's paper [51] claims that any quotient of a subspace of $C_p \oplus_p R_p$ cb-embeds in $L_p(\mathcal{A})$ for some sufficiently large von Neumann algebra $\mathcal{A}$ whenever $1 \le p < 2$. In particular, if $1 \le p < q \le 2$, both R_q and C_q embed completely isomorphically in $L_p(\mathcal{A})$ since both are in $\mathcal{QS}(C_p \oplus_p R_p)$. The last assertion follows as in Lemma 1.1. More precisely, Xu's construction can be done either with $\mathcal{A}$ being the Araki-Woods quasi-free CAR factor and also with Shlyakhtenko's generalization of it in the free setting [45]. In any case, $\mathcal{A}$ can be chosen to be a QWEP type III_λ factor, $0 < \lambda \le 1$. In our first embedding result of this section, we generalize Xu's embedding.

Theorem 4.4. *If for some $1 \le p \le 2$*

$$(\mathrm{X}_1, \mathrm{X}_2) \in \mathcal{QS}(C_p \oplus_2 \mathrm{OH}) \times \mathcal{QS}(R_p \oplus_2 \mathrm{OH}),$$

there exist a cb-embedding $\mathrm{X}_1 \otimes_h \mathrm{X}_2 \to L_p(\mathcal{A})$, for some QWEP algebra $\mathcal{A}$.

The rest of this paragraph is devoted to the proof of Theorem 4.4, which is formally identical to our proof of Theorem C. In the first part of the proof, we reduce the problem to the particular case where both X_1 and X_2 are quotients over certain (annihilators of) graphs.

Part I of the proof of Theorem 4.4. By injectivity of the Haagerup tensor product, we may assume that $(\mathrm{X}_1, \mathrm{X}_2) \in \mathcal{Q}(C_p \oplus_2 \mathrm{OH}) \times \mathcal{Q}(R_p \oplus_2 \mathrm{OH})$. On the other hand, recalling that $C_p = R_p^* = R_{p'}$ for $1 \le p \le \infty$, we easily obtain from Lemma 4.1 and duality the following cb-isomorphisms

$$\begin{aligned} \mathrm{X}_1 &\simeq_{cb} \mathcal{H}_{11,c_p} \oplus_2 \mathcal{H}_{12,oh} \oplus_2 \Big(\big(\mathcal{K}_{11,c_p} \oplus_2 \mathcal{K}_{12,oh}\big) / \mathit{graph}(\Lambda_1)^\perp \Big), \\ \mathrm{X}_2 &\simeq_{cb} \mathcal{H}_{21,r_p} \oplus_2 \mathcal{H}_{22,oh} \oplus_2 \Big(\big(\mathcal{K}_{21,r_p} \oplus_2 \mathcal{K}_{22,oh}\big) / \mathit{graph}(\Lambda_2)^\perp \Big), \end{aligned}$$

for certain subspaces $\mathcal{H}_{ij}, \mathcal{K}_{ij}$ $(1 \le i,j \le 2)$ of ℓ_2 and

$$\begin{array}{rlcl} \Lambda_1: & \mathcal{K}_{11,c_{p'}} & \to & \mathcal{K}_{12,oh}, \\ \Lambda_2: & \mathcal{K}_{21,r_{p'}} & \to & \mathcal{K}_{22,oh}, \end{array}$$

satisfying the properties stated in Lemma 4.1. Let us set

$$\begin{array}{rcl} \mathcal{Z}_1 & = & \big(\mathcal{K}_{11,c_p} \oplus_2 \mathcal{K}_{12,oh}\big) / graph(\Lambda_1)^\perp, \\ \mathcal{Z}_2 & = & \big(\mathcal{K}_{21,r_p} \oplus_2 \mathcal{K}_{22,oh}\big) / graph(\Lambda_2)^\perp. \end{array}$$

Then, we have the following cb-isometric inclusion

$$\begin{array}{rcl} \mathrm{X}_1 \otimes_h \mathrm{X}_2 & \subset & \mathcal{Z}_1 \otimes_h \mathcal{Z}_2 \\ & \oplus_2 & \mathcal{H}_{11,c_p} \otimes_h \mathrm{X}_2 \\ & \oplus_2 & \mathrm{X}_1 \otimes_h \mathcal{H}_{21,r_p} \\ & \oplus_2 & \mathcal{H}_{12,oh} \otimes_h \mathcal{Z}_2 \\ & \oplus_2 & \mathcal{Z}_1 \otimes_h \mathcal{H}_{22,oh} \\ & \oplus_2 & \mathcal{H}_{12,oh} \otimes_h \mathcal{H}_{22,oh}. \end{array} \tag{4.1}$$

Our reduction argument is quite similar to that in Theorem C. Indeed, according to [51] we know that $\mathrm{OH} \in \mathcal{QS}(C_p \oplus_p R_p)$ and that any element in $\mathcal{QS}(C_p \oplus_p R_p)$ completely embeds in $L_p(\mathcal{A})$ for some QWEP type III factor $\mathcal{A}$. This eliminates the last term in (4.1). The second and third terms embed into $S_p(\mathrm{X}_1)$ and $S_p(\mathrm{X}_2)$ completely isometrically. On the other hand, since $\mathrm{OH} \in \mathcal{QS}(C_p \oplus_p R_p)$ and we have by hypothesis

$$\mathrm{X}_1 \in \mathcal{QS}(C_p \oplus_2 \mathrm{OH}) \quad \text{and} \quad \mathrm{X}_2 \in \mathcal{QS}(R_p \oplus_2 \mathrm{OH}),$$

both X_1 and X_2 are cb-isomorphic to an element in $\mathcal{QS}(C_p \oplus_p R_p)$. Applying Xu's theorem [51] one more time, we may eliminate these terms. Finally, for the fourth and fifth terms on the right of (4.1), we apply Lemma 4.2 and the self-duality of OH to rewrite them as particular cases of the first term $\mathcal{Z}_1 \otimes_h \mathcal{Z}_2$. □

Before continuing with the proof, we need more preparation. The following discretization result might be also well-known. Nevertheless, since we are not aware of any reference for it, we include the proof for the sake of completeness.

Lemma 4.5. *Given $1 \le p \le \infty$ and a closed densely-defined operator $\Lambda : R_p \to \mathrm{OH}$ with dense range in* OH, *there exists a diagonal operator $\mathsf{d}_\lambda = \sum_k \lambda_k e_{kk}$ on ℓ_2 such that, when regarded as a map $\mathsf{d}_\lambda : R_p \to \mathrm{OH}$, we obtain*

$$graph(\mathsf{d}_\lambda) \simeq_{cb} graph(\Lambda).$$

Moreover, the relevant constants in the cb-isomorphism above do not depend on Λ.

Proof. Let us first assume that Λ is positive. Then, since R_p is separable we deduce from spectral calculus [22] that there exists a σ-finite measure space $(\Omega, \mathcal{F}, \mu)$ for which Λ is similar to a multiplication operator

$$M_f : L_2(\Omega) \to L_2(\Omega).$$

Thus we may assume $\Lambda = M_f$. Now, we employ a standard procedure to create a diagonal operator. Given $\delta > 0$, we may approximate the function f by an infinite simple function $g = \sum_k (k\delta) 1_{k\delta < f \le (k+1)\delta}$. Replacing f by g yields a $1+\delta$

cb-isomorphism $graph(M_f) \simeq_{cb} graph(M_g)$. Therefore, defining the measurable sets

$$\Omega_k = \Big\{ w \in \Omega \, \big| \, k\delta < f(w) \le (k+1)\delta \Big\},$$

we have that $L_2(\Omega_k)$ is isomorphic to $\ell_2(n_k)$ with $0 \le n_k = \dim L_2(\Omega_k) \le \infty$. Choosing an orthonormal basis for $L_2(\Omega_k)$, we find that M_g is similar to d_λ where $\lambda_k = k\delta$ with multiplicity n_k. This gives the assertion for positive operators. If Λ is not positive, we consider the polar decomposition $\Lambda = u|\Lambda|$. By extension we may assume that u is a unitary. Thus, we get a cb-isometry $graph(\Lambda) \simeq_{cb} graph(|\Lambda|)$. Thus, the general case can be reduced to the case of positive operators. $\square$

We now proceed as above. Given a sequence $\gamma_1, \gamma_2, \ldots \in \mathbb{R}_+$, the diagonal map d_γ is regarded as the density of a *n.s.s.f.* weight ψ on $\mathcal{B}(\ell_2)$. We also keep the same terminology for $q_n, \psi_n, \mathrm{k}_n, \varphi_n, \ldots$ If $1 \le p \le 2$, we define the space $\mathcal{J}_{p',2}(\psi_n)$ as the subspace

$$\Big\{ \big(d_{\psi_n}^{\frac{1}{2p'}} z d_{\psi_n}^{\frac{1}{2p'}}, d_{\psi_n}^{\frac{1}{2p'}} z d_{\psi_n}^{\frac14}, d_{\psi_n}^{\frac14} z d_{\psi_n}^{\frac{1}{2p'}}, d_{\psi_n}^{\frac14} z d_{\psi_n}^{\frac14} \big) \, \big| \, z \in q_n \mathcal{B}(\ell_2) q_n \Big\}$$

of the direct sum

$$\mathcal{L}_{p'}^n = \big(C_{p'}^n \otimes_h R_{p'}^n \big) \oplus_2 \big(C_{p'}^n \otimes_h \mathrm{OH}_n \big) \oplus_2 \big(\mathrm{OH}_n \otimes_h R_{p'}^n \big) \oplus_2 \big(\mathrm{OH}_n \otimes_h \mathrm{OH}_n \big).$$

In other words, we may regard $\mathcal{J}_{p',2}(\psi_n)$ as an intersection of some weighted forms of the asymmetric Schatten classes (see Remark 3.5) considered above. Now we generalize Lemmas 2.11 and 2.12 to the present setting. Let us consider the dual space $\mathcal{K}_{p,2}(\psi_n) = \mathcal{J}_{p',2}(\psi_n)^*$. We assume as above (without lost of generality) that $\mathrm{k}_n = \sum_{k=1}^n \gamma_k$ is an integer and define $\mathcal{A}_n$ as in Lemma 2.11. If π_j is the natural embedding into the j-th component of $\mathcal{A}_n$ and we set $x_j = \pi_j(x, -x)$, the following result is the L_p version of Lemma 2.11.

Lemma 4.6. *The mapping*

$$\omega : x \in \mathcal{K}_{p,2}(\psi_n) \mapsto \frac{1}{\mathrm{k}_n} \sum_{j=1}^{\mathrm{k}_n} x_j \otimes \delta_j \in L_p(\mathcal{A}_n; \mathrm{OH}_{\mathrm{k}_n})$$

is a cb-embedding with cb-complemented image and constants independent of n.

Proof. The complete isometry $\mathcal{J}_{p',2}(\psi_n) = \mathcal{J}_{p',2}^{\mathrm{k}_n}(q_n \mathcal{B}(\ell_2) q_n)$ is all what we need since the argument is completed as in Lemma 2.11 replacing the use of Corollary 2.9 by its generalized form given in Corollary 3.10. On the other hand, we may rewrite the space $\mathcal{J}_{p',2}(\psi_n)$ using the language of conditional L_p spaces. Indeed, let us consider the von Neumann algebra $\mathcal{M}_n = q_n \mathcal{B}(\ell_2) q_n$ equipped with the state φ_n which arises from the relation $\psi_n = \mathrm{k}_n \varphi_n$. Then the space $\mathcal{J}_{p',2}(\psi_n)$ has the form (see Remark 3.5)

$$\mathrm{k}_n^{\frac{1}{p'}} L_{p'}(\mathcal{M}_n) \cap \mathrm{k}_n^{\frac{1}{2p'} + \frac14} L_{(2p',4)}(\mathcal{M}_n) \cap \mathrm{k}_n^{\frac14 + \frac{1}{2p'}} L_{(4,2p')}(\mathcal{M}_n) \cap \mathrm{k}_n^{\frac12} L_{(4,4)}(\mathcal{M}_n).$$

This is exactly the definition of $\mathcal{J}_{p',2}^{\mathrm{k}_n}(q_n \mathcal{B}(\ell_2) q_n)$ and the proof is complete. $\square$

Lemma 4.7. *If $\lambda_1, \lambda_2, \ldots \in \mathbb{R}_+$, we set*

$$\begin{aligned} R_{p'} \cap \ell_2^{oh}(\lambda) &= \mathrm{span} \Big\{ (\delta_k, \lambda_k \delta_k) \in R_{p'} \oplus_2 \mathrm{OH} \Big\}, \\ C_{p'} \cap \ell_2^{oh}(\lambda) &= \mathrm{span} \Big\{ (\delta_k, \lambda_k \delta_k) \in C_{p'} \oplus_2 \mathrm{OH} \Big\}, \end{aligned}$$

$$\begin{aligned} R_p + \ell_2^{oh}(\lambda) &= \big(R_p \oplus_2 \mathrm{OH}\big) / \big(R_{p'} \cap \ell_2^{oh}(\lambda)\big)^{\perp}, \\ C_p + \ell_2^{oh}(\lambda) &= \big(C_p \oplus_2 \mathrm{OH}\big) / \big(C_{p'} \cap \ell_2^{oh}(\lambda)\big)^{\perp}. \end{aligned}$$

Then, there exists a n.s.s.f. weight ψ on $\mathcal{B}(\ell_2)$ such that

$$\big(C_p + \ell_2^{oh}(\lambda)\big) \otimes_h \big(R_p + \ell_2^{oh}(\lambda)\big) = \mathcal{K}_{p,2}(\psi) = \overline{\bigcup_{n\ge1} \mathcal{K}_{p,2}(\psi_n)}.$$

Proof. Let us define

$$\begin{aligned} q_n\big(R_p + \ell_2^{oh}(\lambda)\big) &= \Big\{ (q_n(a), q_n(b)) + \big(R_{p'} \cap \ell_2^{oh}(\lambda)\big)^{\perp} \, \big| \, (a,b) \in R_p \oplus_2 \mathrm{OH} \Big\}, \\ q_n\big(C_p + \ell_2^{oh}(\lambda)\big) &= \Big\{ (q_n(a), q_n(b)) + \big(C_{p'} \cap \ell_2^{oh}(\lambda)\big)^{\perp} \, \big| \, (a,b) \in C_p \oplus_2 \mathrm{OH} \Big\}. \end{aligned}$$

Arguing as in the proof of Lemma 2.12 ($q_n(x) \to x$ as $n \to \infty$ in R_p, OH, C_p), we may write the Haagerup tensor product $\big(C_p + \ell_2^{oh}(\lambda)\big) \otimes_h \big(R_p + \ell_2^{oh}(\lambda)\big)$ as the direct limit below

$$\overline{\bigcup_{n\ge1} q_n(C_p + \ell_2^{oh}(\lambda)) \otimes_h q_n(R_p + \ell_2^{oh}(\lambda))}.$$

This reduces the problem to the finite-dimensional case. Arguing by duality, we have to show that $q_n(C_{p'} \cap \ell_2^{oh}(\lambda)) \otimes_h q_n(R_{p'} \cap \ell_2^{oh}(\lambda)) = \mathcal{J}_{p',2}(\psi_n)$ for some *n.s.s.f.* weight ψ, where

$$\begin{aligned} q_n\big(C_{p'} \cap \ell_2^{oh}(\lambda)\big) &= \mathrm{span}\Big\{ (e_{i1}, \lambda_i e_{i1}) \in C_{p'}^n \oplus_2 \mathrm{OH}_n \Big\}, \\ q_n\big(R_{p'} \cap \ell_2^{oh}(\lambda)\big) &= \mathrm{span}\Big\{ (e_{1j}, \lambda_j e_{1j}) \in R_{p'}^n \oplus_2 \mathrm{OH}_n \Big\}. \end{aligned}$$

Its Haagerup tensor product is the subspace

$$\mathrm{span}\Big\{ (e_{ij}, \lambda_j e_{ij}, \lambda_i e_{ij}, \lambda_i \lambda_j e_{ij}) \Big\} = \Big\{ (x, x\mathsf{d}_\lambda, \mathsf{d}_\lambda x, \mathsf{d}_\lambda x \mathsf{d}_\lambda) \, \big| \, x \in q_n \mathcal{B}(\ell_2) q_n \Big\}$$

of the space $\mathcal{L}_{p'}^n$ defined above. Then, we define $\gamma_k \in \mathbb{R}_+$ by the relation

$$\lambda_k = \gamma_k^{\frac14 - \frac{1}{2p'}} \quad \text{and} \quad z = d_{\psi_n}^{-\frac{1}{2p'}} x d_{\psi_n}^{-\frac{1}{2p'}},$$

where ψ is the *n.s.s.f.* weight induced by d_γ. This gives the space

$$q_n(\mathcal{G}_{p'}^c(\lambda)) \otimes_h q_n(\mathcal{G}_{p'}^r(\lambda)) = \Big\{ \big(d_{\psi_n}^{\frac{1}{2p'}} z d_{\psi_n}^{\frac{1}{2p'}}, d_{\psi_n}^{\frac{1}{2p'}} z d_{\psi_n}^{\frac14}, d_{\psi_n}^{\frac14} z d_{\psi_n}^{\frac{1}{2p'}}, d_{\psi_n}^{\frac14} z d_{\psi_n}^{\frac14} \big) \Big\}.$$

The space on the right is by definition $\mathcal{J}_{p',2}(\psi_n)$. This completes the proof. $\square$

Part II of the proof of Theorem 4.4. By Lemma 4.5 we may assume that the graphs appearing in the terms $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are graphs of diagonal operators d_{λ_1} and d_{λ_2}. By polar decomposition, perturbation and complementation (as in the proof of Theorem C via Lemmas 4.2 and 4.5), we may assume that

$$\begin{aligned} \mathcal{Z}_1 &= C_p + \ell_2^{oh}(\lambda), \\ \mathcal{Z}_2 &= R_p + \ell_2^{oh}(\lambda), \end{aligned}$$

with $\lambda_1, \lambda_2, \ldots \in \mathbb{R}_+$ strictly positive. According to Lemma 4.7, we conclude that $\mathcal{Z}_1 \otimes_h \mathcal{Z}_2$ can be identified with $\mathcal{K}_{p,2}(\psi)$ for some *n.s.s.f.* weight ψ on $\mathcal{B}(\ell_2)$. It remains to construct a complete embedding of $\mathcal{K}_{p,2}(\psi)$ into $L_p(\mathcal{A})$ for some QWEP algebra $\mathcal{A}$. To that aim, we assume without loss of generality that the k_n's are

integers. This allows us to proceed in the usual way. Namely, we first embed $\mathcal{K}_{p,2}(\psi)$ into an ultraproduct

$$\mathcal{K}_{p,2}(\psi) = \overline{\bigcup_{n\geq 1} \mathcal{K}_{p,2}(\psi_n)} \to \prod\nolimits_{n,\mathcal{U}} \mathcal{K}_{p,2}(\psi_n).$$

According to [41], this reduces the problem to the finite-dimensional case, which follows from Xu's cb-embedding [51] of OH into $L_p(\mathcal{B})$ for some QWEP type III factor $\mathcal{B}$ and from Lemma 4.6

$$\mathcal{K}_{p,2}(\psi_n) \to L_p(\mathcal{A}_n; \mathrm{OH}_{\mathrm{k}_n}) \to L_p(\mathcal{A}_n \bar{\otimes} \mathcal{B}) = L_p(\mathcal{A}'_n).$$

We have therefore constructed a cb-embedding

$$\mathcal{Z}_1 \otimes_h \mathcal{Z}_2 \to L_p(\mathcal{A}) \quad \text{with} \quad \mathcal{A} = \Big(\prod\nolimits_{n,\mathcal{U}} \mathcal{A}'_{n*} \Big)^*.$$

The fact that $\mathcal{A}$ is QWEP is justified as in the proof of Theorem C. □

Corollary 4.8. *S_q cb-embeds into $L_p(\mathcal{A})$ for some* QWEP *algebra $\mathcal{A}$.*

Proof. Since $S_q = C_q \otimes_h R_q$, it follows from Lemma 1.1 and Theorem 4.4. □

4.2. **Embedding into the hyperfinite factor.** Now we want to show that the cb-embedding $S_q \to L_p(\mathcal{A})$ can be constructed with $\mathcal{A}$ being a hyperfinite type III factor. Moreover, we shall prove some more general results to be used in the next paragraph, where the general cb-embedding will be constructed. We first establish a transference argument, based on a noncommutative form of Rosenthal's inequality for identically distributed random variables in L_1 from [13], which enables us to replace freeness by some sort of independence.

Let $\mathcal{N}$ be a σ-finite von Neumann subalgebra of some algebra $\mathcal{A}$ and let us consider a family $\mathsf{A}_1, \mathsf{A}_2, \ldots$ of von Neumann algebras with $\mathcal{N} \subset \mathsf{A}_k \subset \mathcal{A}$. As usual, we require the existence of a *n.f.* conditional expectation $\mathsf{E}_\mathcal{N} : \mathcal{A} \to \mathcal{N}$. We recall that $(\mathsf{A}_k)_{k\geq 1}$ is a system of *indiscernible independent copies over $\mathcal{N}$* (*i.i.c.* in short) when

- **(i)** If $a \in \big\langle \mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_{k-1} \big\rangle$ and $b \in \mathsf{A}_k$, we have
$$\mathsf{E}_\mathcal{N}(ab) = \mathsf{E}_\mathcal{N}(a)\mathsf{E}_\mathcal{N}(b).$$
- **(ii)** There exist a von Neumann algebra A containing $\mathcal{N}$, a normal faithful conditional expectation $\mathsf{E}_0 : \mathsf{A} \to \mathcal{N}$ and homomorphisms $\pi_k : \mathsf{A} \to \mathsf{A}_k$ such that
$$\mathsf{E}_\mathcal{N} \circ \pi_k = \mathsf{E}_0$$
and the following holds for every strictly increasing function $\alpha : \mathbb{N} \to \mathbb{N}$
$$\mathsf{E}_\mathcal{N}\big(\pi_{j_1}(a_1) \cdots \pi_{j_m}(a_m)\big) = \mathsf{E}_\mathcal{N}\big(\pi_{\alpha(j_1)}(a_1) \cdots \pi_{\alpha(j_m)}(a_m)\big).$$
- **(iii)** There exist *n.f.* conditional expectations $\mathcal{E}_k : \mathcal{A} \to \mathsf{A}_k$ such that
$$\mathsf{E}_\mathcal{N} = \mathsf{E}_0 \pi_k^{-1} \mathcal{E}_k \quad \text{for all} \quad k \geq 1.$$

We shall say that $(\mathsf{A}_k)_{k\geq 1}$ are *symmetrically independent copies over $\mathcal{N}$* (*s.i.c.* in short) when the first condition above also holds for a in the algebra generated by $\mathsf{A}_1, \ldots, \mathsf{A}_{k-1}, \mathsf{A}_{k+1}, \ldots$ and the second condition holds for any permutation α of the integers. In what follows, given a probability space (Ω, μ), we shall write $\varepsilon_1, \varepsilon_2, \ldots$ to denote an independent family of Bernoulli random variables on Ω equidistributed on ± 1. We now present the key inequality in [13].

Lemma 4.9. *The following inequalities hold for* $x \in L_1(\mathsf{A})$:

a) *If* $(\mathsf{A}_k)_{k\ge1}$ *are i.i.c. over* $\mathcal{N}$, *we have*

$$\int_\Omega \Big\| \sum_{k=1}^n \varepsilon_k \pi_k(x) \Big\|_{L_1(\mathcal{A})} d\mu$$
$$\sim \inf_{x=a+b+c} n\|a\|_{L_1(\mathsf{A})} + \sqrt{n}\big\|\mathsf{E}_0(bb^*)^{\frac12}\big\|_{L_1(\mathcal{N})} + \sqrt{n}\big\|\mathsf{E}_0(c^*c)^{\frac12}\big\|_{L_1(\mathcal{N})}.$$

b) *If moreover,* $\mathsf{E}_0(x)=0$ *and* $(\mathsf{A}_k)_{k\ge1}$ *are s.i.c. over* $\mathcal{N}$, *then*

$$\Big\| \sum_{k=1}^n \pi_k(x) \Big\|_{L_1(\mathcal{A})}$$
$$\sim \inf_{x=a+b+c} n\|a\|_{L_1(\mathsf{A})} + \sqrt{n}\big\|\mathsf{E}_0(bb^*)^{\frac12}\big\|_{L_1(\mathcal{N})} + \sqrt{n}\big\|\mathsf{E}_0(c^*c)^{\frac12}\big\|_{L_1(\mathcal{N})}.$$

Proof. We claim that

$$\frac12\Big\| \sum_{k=1}^n \pi_k(x) \Big\|_{L_1(\mathcal{A})} \le \Big\| \sum_{k=1}^n \varepsilon_k\pi_k(x) \Big\|_{L_1(\mathcal{A})} \le 2\Big\| \sum_{k=1}^n \pi_k(x) \Big\|_{L_1(\mathcal{A})}$$

for any choice of signs $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n$ whenever $\mathsf{A}_1,\mathsf{A}_2,\ldots$ are symmetric independent copies of A over $\mathcal{N}$ and $\mathsf{E}_0(x)=0$. This establishes (a) $\Rightarrow$ (b) and so, since the first assertion is proved in [13], it suffices to prove our claim. Such result will follow from the more general statement

$$\Big\| \sum_{k=1}^n \varepsilon_k\pi_k(x_k) \Big\|_{L_1(\mathcal{A})} \le 2\Big\| \sum_{k=1}^n \pi_k(x_k) \Big\|_{L_1(\mathcal{A})},$$

for any family $x_1,x_2,\ldots,x_n$ in $L_1(\mathsf{A})$ with $\mathsf{E}_0(x_k)=0$ for $1\le k\le n$. Since we assume that $\mathcal{N}$ is σ-finite we may fix a *n.f.* state φ. We define $\phi=\varphi\circ\mathsf{E}_{\mathcal{N}}$ and $\phi_0=\varphi\circ\mathsf{E}_0$. According to [3] we have

$$\sigma_t^\varphi\circ\mathsf{E}_0=\mathsf{E}_0\circ\sigma_t^{\phi_0} \quad \text{and} \quad \sigma_t^\varphi\circ\mathsf{E}_{\mathcal{N}}=\mathsf{E}_{\mathcal{N}}\circ\sigma_t^{\phi}.$$

Moreover, since $\mathsf{E}_{\mathcal{N}}=\mathsf{E}_{\mathcal{N}}\circ\mathcal{E}_k$ we find $\phi=\phi\circ\mathcal{E}_k$ which implies

$$\sigma_t^\phi\circ\mathcal{E}_k=\mathcal{E}_k\circ\sigma_t^\phi.$$

In particular, $\sigma_t^\phi(\mathsf{A}_k)\subset\mathsf{A}_k$ for $k\ge1$. Therefore, given any subset S of $\{1,2,\ldots,n\}$ we find a ϕ-invariant conditional expectation $\mathsf{E}_\mathsf{S}:\mathcal{A}\to\mathcal{A}_\mathsf{S}$ where the von Neumann algebra $\mathcal{A}_\mathsf{S}=\langle\mathsf{A}_k \mid k\in\mathsf{S}\rangle$. We claim that

$$\mathsf{E}_\mathsf{S}(\pi_j(a))=0 \quad \text{whenever} \quad \mathsf{E}_0(a)=0 \quad \text{and} \quad j\notin\mathsf{S}.$$

Indeed, let $b\in\mathcal{A}_\mathsf{S}$ and a as above. Then we deduce from symmetric independence

$$\phi\big(\mathsf{E}_\mathsf{S}(\pi_j(a))b\big)=\phi\big(\pi_j(a)b\big)=\phi\big(\mathsf{E}_{\mathcal{N}}(\pi_j(a)b)\big)=\phi\big(\mathsf{E}_0(a)\mathsf{E}_{\mathcal{N}}(b)\big)=0.$$

Thus we may apply Doob's trick

$$\Big\| \sum_{k\in\mathsf{S}} \pi_k(x_k) \Big\|_{L_1(\mathcal{A})} = \Big\| \mathsf{E}_\mathsf{S}\Big(\sum_{k=1}^n \pi_k(x_k)\Big) \Big\|_{L_1(\mathcal{A})} \le \Big\| \sum_{k=1}^n \pi_k(x_k) \Big\|_{L_1(\mathcal{A})}.$$

Then, the claim follows taking $\{1,2,\ldots,n\}=\mathsf{S}_1\cup\mathsf{S}_{-1}$ with $\mathsf{S}_\alpha=\{k:\varepsilon_k=\alpha\}$. $\square$

Remark 4.10. The inequalities in Lemma 4.9 generalize the noncommutative Rosenthal inequality [21] to the case $p = 1$ for identically distributed variables and under such notions of noncommutative independence. Of course, in the case $1 < p < 2$ we have much stronger results from [19, 21] and there is no need of proving any preliminary result for our aims in this case.

Let us now generalize our previous definition of the space $\mathcal{K}_{p,2}(\psi)$ to general von Neumann algebras. Let $\mathcal{M}$ be a given von Neumann algebra, which we assume σ-finite for the sake of clarity. Let $\mathcal{M}$ be equipped with a *n.s.s.f.* weight ψ. In other words, ψ is given by an increasing sequence (a net in the general case) of pairs (ψ_n, q_n) such that the q_n's are increasing finite projections in $\mathcal{M}$ with $\lim_n q_n = 1$ in the strong operator topology and $\sigma_t^{\psi}(q_n) = q_n$. Moreover, the ψ_n's are normal positive functionals on $\mathcal{M}$ with support q_n and satisfying the compatibility condition $\psi_{n+1}(q_n x q_n) = \psi_n(x)$. As above, we shall write k_n for the number $\psi_n(q_n) \in (0,\infty)$ and (again as above) we may and will assume that the k_n's are nondecreasing positive integers. In what follows we shall write d_{ψ_n} for the density on $q_n\mathcal{M}q_n$ associated to the *n.f.* finite weight ψ_n. If $1 \le p \le 2$, we define the space $\mathcal{J}_{p',2}(\psi_n)$ as the closure of

$$\Big\{ \big(d_{\psi_n}^{\frac{1}{2p'}} z d_{\psi_n}^{\frac{1}{2p'}}, d_{\psi_n}^{\frac{1}{2p'}} z d_{\psi_n}^{\frac14}, d_{\psi_n}^{\frac14} z d_{\psi_n}^{\frac{1}{2p'}}, d_{\psi_n}^{\frac14} z d_{\psi_n}^{\frac14} \big) \ \big| \ z \in q_n\mathcal{M}q_n \Big\}$$

in the direct sum

$$\mathcal{L}_{p'}^n = L_{p'}(q_n\mathcal{M}q_n) \oplus_2 L_{(2p',4)}(q_n\mathcal{M}q_n) \oplus_2 L_{(4,2p')}(q_n\mathcal{M}q_n) \oplus_2 L_2(q_n\mathcal{M}q_n).$$

In other words, after considering the *n.f.* state φ_n on $q_n\mathcal{M}q_n$ determined by the relation $\psi_n = \mathrm{k}_n\varphi_n$ and recalling the definition of the spaces $\mathcal{J}_{p,q}^n(\mathcal{M})$ from Section 3, we may regard $\mathcal{J}_{p',2}(\psi_n)$ as the 4-term intersection space

$$\mathcal{J}_{p',2}(\psi_n) = \bigcap_{u,v \in \{2p',4\}} \mathrm{k}_n^{\frac1u + \frac1v} L_{(u,v)}(q_n\mathcal{M}q_n) = \mathcal{J}_{p',2}^{\mathrm{k}_n}(q_n\mathcal{M}q_n).$$

Now we take direct limits and define

$$\mathcal{J}_{p',2}(\psi) = \overline{\bigcup_{n\ge1} \mathcal{J}_{p',2}(\psi_n)},$$

where the closure is taken with respect to the norm of the space

$$\mathcal{L}_{p'} = L_{p'}(\mathcal{M}) \oplus_2 L_{(2p',4)}(\mathcal{M}) \oplus_2 L_{(4,2p')}(\mathcal{M}) \oplus_2 L_2(\mathcal{M}).$$

To define the space $\mathcal{K}_{p,2}(\psi)$ we also proceed as above and consider

$$\Psi_n : \mathcal{L}_p^n \to L_1(q_n\mathcal{M}q_n)$$

given by

$$\Psi_n(x_1,x_2,x_3,x_4) = d_{\psi_n}^{\frac{1}{2p'}} x_1 d_{\psi_n}^{\frac{1}{2p'}} + d_{\psi_n}^{\frac{1}{2p'}} x_2 d_{\psi_n}^{\frac14} + d_{\psi_n}^{\frac14} x_3 d_{\psi_n}^{\frac{1}{2p'}} + d_{\psi_n}^{\frac14} x_4 d_{\psi_n}^{\frac14}.$$

This gives $\ker \Psi_n = \mathcal{J}_{p',2}(\psi_n)^\perp$ and we define

$$\mathcal{K}_{p,2}(\psi_n) = \mathcal{L}_p^n / \ker \Psi_n \quad \text{and} \quad \mathcal{K}_{p,2}(\psi) = \overline{\bigcup_{n\ge1} \mathcal{K}_{p,2}(\psi_n)},$$

where the latter is understood as a quotient of $\mathcal{L}_p$. In other words, we may regard the space $\mathcal{K}_{p,2}(\psi_n)$ as the sum of the corresponding dual weighted asymmetric L_p spaces considered in the definition of $\mathcal{J}_{p',2}(\psi_n)$

$$\mathcal{K}_{p,2}(\psi_n) = \sum_{u,v \in \{2p,4\}} \mathrm{k}_n^{\frac{1}{u}+\frac{1}{v}} L_{(u,v)}(q_n \mathcal{M} q_n). \tag{4.2}$$

Thus, using $\psi_n = \mathrm{k}_n \varphi_n$ backwards and taking direct limits

$$\mathcal{K}_{p,2}(\psi) = L_p(\mathcal{M}) + L_{(2p,4)}(\mathcal{M}) + L_{(4,2p)}(\mathcal{M}) + L_2(\mathcal{M}),$$

where the sum is taken in $L_p(\mathcal{M})$ and the embeddings are given by

$$\begin{array}{rcl} j_c(x) : x \in L_{(2p,4)}(\mathcal{M}) & \mapsto & x d_\psi^\beta \in L_p(\mathcal{M}), \\ j_r(x) : x \in L_{(4,2p)}(\mathcal{M}) & \mapsto & d_\psi^\beta x \in L_p(\mathcal{M}), \end{array}$$

with $\beta = 1/2p - 1/4$, while the embedding of $L_2(\mathcal{M})$ into $L_p(\mathcal{M})$ is given by

$$j_2(x) = d_\psi^\beta x d_\psi^\beta.$$

Remark 4.11. It will be important below to observe that our definition of $\mathcal{K}_{p,2}(\psi_n)$ is slightly different to the one given in the previous paragraph. Indeed, according to the usual duality bracket $\langle x, y \rangle = \mathrm{tr}(x^* y)$, we should have defined

$$\mathcal{K}_{p,2}(\psi_n) = \sum_{u,v \in \{2p,4\}} \mathrm{k}_n^{-\gamma(u,v)} L_{(u,v)}(q_n \mathcal{M} q_n) \quad \text{with} \quad \gamma(u,v) = \frac{1}{2(u/2)'} + \frac{1}{2(v/2)'}.$$

This would give $\mathcal{K}_{p,2}(\psi_n) = \mathcal{J}_{p',2}(\psi_n)^*$ and

$$\mathcal{K}_{p,2}(\psi_n) = \frac{1}{\mathrm{k}_n} \sum_{u,v \in \{2p,4\}} \mathrm{k}_n^{\frac{1}{u}+\frac{1}{v}} L_{(u,v)}(q_n \mathcal{M} q_n).$$

However, we prefer to use (4.2) in what follows for notational convenience.

Now we set some notation to distinguish between independent and free random variables. If we fix a positive integer n, the von Neumann algebra $\mathcal{A}_{ind}^n$ will denote the k_n-fold tensor product of $q_n \mathcal{M} q_n$ while $\mathcal{A}_{free}^n$ will be (as usual) the k_n-fold free product of $q_n \mathcal{M} q_n \oplus q_n \mathcal{M} q_n$. In other words, if we set

$$\widetilde{\mathsf{A}}_{n,j} = q_n \mathcal{M} q_n \quad \text{and} \quad \mathsf{A}_{n,j} = q_n \mathcal{M} q_n \oplus q_n \mathcal{M} q_n$$

for $1 \le j \le \mathrm{k}_n$, we define the following von Neumann algebras

$$\begin{array}{rcl} \mathcal{A}_{ind}^n & = & \otimes_j \widetilde{\mathsf{A}}_{n,j}, \\ \mathcal{A}_{free}^n & = & *_j \, \mathsf{A}_{n,j}. \end{array}$$

We also consider the natural embeddings

$$\pi_{ind}^j : \widetilde{\mathsf{A}}_{n,j} \to \mathcal{A}_{ind}^n \quad \text{and} \quad \pi_{free}^j : \mathsf{A}_{n,j} \to \mathcal{A}_{free}^n.$$

We need some further information on OH. Given $1 < p < 2$, Xu constructed in [51] a complete embedding of OH into $L_p(\mathcal{A})$ with $\mathcal{A}$ hyperfinite, while for $p = 1$ the corresponding cb-embedding was recently constructed in [13]. The argument can be sketched with the following chain

$$\mathrm{OH} \hookrightarrow \big(C_p \oplus_p R_p\big)/graph(\mathsf{d}_\lambda)^\perp \simeq_{cb} C_p + R_p(\lambda) \hookrightarrow L_p(\mathcal{A}).$$

Indeed, arguing as in Lemma 1.1/Remark 1.3 and applying Lemma 4.5, we see how to regard OH as a subspace of a quotient of $C_p \oplus_p R_p$ by the annihilator of

some diagonal map $\mathsf{d}_\lambda : C_{p'} \to R_{p'}$. By the action of d_λ, the annihilator of its graph is the span of elements of the form $(\delta_k, -\delta_k/\lambda_k)$. This suggest to regard the quotient above as the sum of C_p with a weighted form of R_p. This establishes the cb-isomorphism in the middle. Then, it is natural to guess that the complete embedding into $L_p(\mathcal{A})$ should follow from a *weighted* form of the noncommutative Khintchine inequality. The first inequality of this kind was given by Pisier and Shlyakhtenko in [37] for generalized circular variables and further investigated in [18, 50]. However, if we want to end up with a hyperfinite von Neumann algebra $\mathcal{A}$, we must replace generalized circulars by their Fermionic analogues. More precisely, given a complex Hilbert space $\mathcal{H}$, we consider its antisymmetric Fock space $\mathcal{F}_{-1}(\mathcal{H})$. Let $c(e)$ and $a(e)$ denote the creation and annihilation operators associated with a vector $e \in \mathcal{H}$. Given an orthonormal basis $(e_{\pm k})_{k\geq 1}$ of $\mathcal{H}$ and a family $(\mu_k)_{k\geq 1}$ of positive numbers, we set $f_k = c(e_k) + \mu_k\, a(e_{-k})$. The sequence $(f_k)_{k\geq 1}$ satisfies the canonical anticommutation relations and we take $\mathcal{A}$ to be the von Neumann algebra generated by the f_k's. Taking suitable μ_k's depending only on p and the eigenvalues of d_λ, the Khintchine inequality associated to the system of f_k's provides the desired cb-embedding. Namely, let ϕ be the quasi-free state on $\mathcal{A}$ determined by the vacuum and let d_ϕ be the associated density. Then, if $(\delta_k)_{k\geq 1}$ denotes the unit vector basis of OH, the cb-embedding has the form

$$w(\delta_k) = \xi_k\, d_\phi^{\frac{1}{2p}}\, f_k\, d_\phi^{\frac{1}{2p}} = \xi_k\, f_{p,k}$$

for some scaling factors $(\xi_k)_{k\geq 1}$. The necessary Khintchine type inequalities for $1 < p < 2$ follow from the noncommutative Burkholder inequality [19]. In the L_1 case, the key inequalities follow from Lemma 4.9, see [13] for details. With this construction, the von Neumann algebra $\mathcal{A}$ turns out to be the Araki-Woods factor arising from the GNS construction applied to the CAR algebra with respect to the quasi-free state ϕ. In fact, using a conditional expectation, we can replace the μ_k's by a sequence $(\mu_k')_{k\geq 1}$ such that for every rational $0 < \lambda < 1$ there are infinitely many μ_k''s with $\mu_k' = \lambda/(1+\lambda)$. According to the results in [1], we then obtain the hyperfinite type III_1 factor $\mathcal{R}$.

On the other hand, there exists a slight modification of this construction which will be used below. Indeed, using the terminology introduced above and following [13] there exists a mean-zero $\gamma_p \in L_p(\mathcal{R})$ given by a linear combination of the $f_{p,k}$'s such that

$$w(\delta_j) = \pi_{ind}^j(\gamma_p)$$

defines a completely isomorphic embedding

$$w : \mathrm{OH}_{\mathrm{k}_n} \to L_p(\mathcal{R}_{\otimes \mathrm{k}_n})$$

with constants independent of n. Here $\mathcal{R}_{\otimes \mathrm{k}_n}$ denotes the k_n-fold tensor product of $\mathcal{R}$. Moreover, given any von Neumann algebra $\mathcal{A}$, $id_{L_p(\mathcal{A})} \otimes w$ also defines an isomorphism

$$id_{L_p(\mathcal{A})} \otimes w : L_p\big(\mathcal{A}; \mathrm{OH}_{\mathrm{k}_n}\big) \to L_p(\mathcal{A} \bar{\otimes} \mathcal{R}_{\otimes \mathrm{k}_n}).$$

Proposition 4.12. *If $1 \leq p \leq 2$, the map*

$$\xi_{ind}^n : x \in \mathcal{K}_{p,2}(\psi_n) \mapsto \sum_{j=1}^{\mathrm{k}_n} \pi_{ind}^j(x) \otimes \delta_j \in L_p\big(\mathcal{A}_{ind}^n; \mathrm{OH}_{\mathrm{k}_n}\big)$$

is a completely isomorphic embedding with relevant constants independent of n.

Proof. According to Corollary 3.10, this is true for

$$\xi_{free}^n : x \in \mathcal{K}_{p,2}(\psi_n) \mapsto \sum_{j=1}^{\mathrm{k}_n} \pi_{free}^j(x,-x) \otimes \delta_j \in L_p\big(\mathcal{A}_{free}^n; \mathrm{OH}_{\mathrm{k}_n}\big).$$

Indeed, it follows from a simple duality argument (see [15] or Remark 7.4 [16]) taking Remark 4.11 into account. According to the preceding discussion on OH, we deduce that

$$w \circ \xi_{free}^n : x \in \mathcal{K}_{p,2}(\psi_n) \mapsto \sum_{j=1}^{n} \pi_{free}^j(x,-x) \otimes \pi_{ind}^j(\gamma_p) \in L_p\big(\mathcal{A}_{free}^n \bar{\otimes} \mathcal{R}_{\otimes \mathrm{k}_n}\big)$$

also provides a cb-isomorphism. Now, we consider

$$\begin{array}{rcl} \widetilde{\mathsf{B}}_{n,j} & = & \pi_{ind}^j(\widetilde{\mathsf{A}}_{n,j}) \otimes \pi_{ind}^j(\mathcal{R}), \\ \mathsf{B}_{n,j} & = & \pi_{free}^j(\mathsf{A}_{n,j}) \otimes \pi_{ind}^j(\mathcal{R}), \end{array}$$

for $1 \le j \le \mathrm{k}_n$. It is clear from the construction that both families of von Neumann algebras are *s.i.c.* over the complex field. Therefore, Lemma 4.9/Remark 4.10 apply in both cases (note that the mean-zero condition for the $\widetilde{\mathsf{B}}_{n,j}$'s holds due to the fact that γ_p is mean-zero) and hence

$$\big\| id_{S_p^m} \otimes w \circ \xi_{free}^n(x) \big\|_p \sim_c \big\| id_{S_p^m} \otimes w \circ \xi_{ind}^n(x) \big\|_p$$

holds for every element $x \in L_p\big(\mathrm{M}_m(q_n \mathcal{M} q_n)\big)$. This completes the proof. $\square$

Remark 4.13. Proposition 4.12 can be regarded as a generalization of Lemma 4.6 for general von Neumann algebras, where freeness is replaced by noncommutative independence. Indeed, the only difference between both results is the factor $1/\mathrm{k}_n$, which is explained as a byproduct of Remark 4.11.

Remark 4.14. The transference argument applied in the proof of Proposition 4.12 gives a result which might be of independent interest. Given a von Neumann algebra A, let us construct the tensor product $\mathcal{A}_{ind}$ of infinitely many copies of $\mathsf{A} \oplus \mathsf{A}$. Similarly, the free product $\mathcal{A}_{free}$ of infinitely many copies of $\mathsf{A} \oplus \mathsf{A}$ will be considered. Following our terminology, we have maps

$$\pi_{ind}^j : \mathsf{A} \to \mathcal{A}_{ind} \quad \text{and} \quad \pi_{free}^j : \mathsf{A} \to \mathcal{A}_{free}.$$

If $1 < p < q < 2$, we claim that

$$\Big\| \sum\nolimits_j \pi_{ind}^j(x,-x) \otimes \delta_j \Big\|_{L_p(\mathcal{A}_{ind};\ell_q)} \sim_{cb} \Big\| \sum\nolimits_j \pi_{free}^j(x,-x) \otimes \delta_j \Big\|_{L_p(\mathcal{A}_{free};\ell_q)}, \tag{4.3}$$

where the symbol $\sim_{cb}$ is used to mean that the equivalence also holds (with absolute constants) when taking the matrix norms arising from the natural operator space structures of the spaces considered. The case $q = 2$ follows by using exactly the same argument as in Proposition 4.12. In fact, the same idea works for general indices. Indeed, we just need to embed ℓ_q into L_p completely isometrically and then use the noncommutative Rosenthal inequality [21]. Recall that the cb-embedding of ℓ_q into L_p is already known at this stage of the paper as a consequence of Corollary 4.8. At the time of this writing, it is still open whether or not (4.3) is still valid for other values of (p, q).

Our main goal in this paragraph is to generalize the complete embedding in Proposition 4.12 to the direct limit $\mathcal{K}_{p,2}(\psi)$. Of course, this is possible using an ultraproduct procedure. However, this would not preserve hyperfiniteness. We will now explain how the proof of Proposition 4.12 allows to factorize the cb-embedding $\mathcal{K}_{p,2}(\psi_n) \to L_p(\mathcal{A}_{ind}^n \bar{\otimes} \mathcal{R}_{\otimes \mathrm{k}_n})$ via a three term K-functional. We will combine this with the concept of noncommutative Poisson random measure from [14] to produce a complete embedding which preserves the direct limit mentioned above. Let us consider the operator space

$$\mathcal{K}_{rc_p}^p(\psi_n) = \mathrm{k}_n^{\frac{1}{p}} L_p(q_n \mathcal{M} q_n) + \mathrm{k}_n^{\frac{1}{2}} L_2^{r_p}(q_n \mathcal{M} q_n) + \mathrm{k}_n^{\frac{1}{2}} L_2^{c_p}(q_n \mathcal{M} q_n),$$

where the norms in the L_p spaces considered above are calculated with respect to the state φ_n arising from the relation $\psi_n = \mathrm{k}_n \varphi_n$. More precisely, the operator space structure is determined by

$$\|x\|_{S_p^m(\mathcal{K}_{rc_p}^p(\psi_n))} = \inf \Big\{ \mathrm{k}_n^{\frac{1}{p}} \|x_1\|_{S_p^m(L_p)} + \mathrm{k}_n^{\frac{1}{2}} \|x_2\|_{S_p^m(L_2^{r_p})} + \mathrm{k}_n^{\frac{1}{2}} \|x_3\|_{S_p^m(L_2^{c_p})} \Big\},$$

where the infimum runs over all possible decompositions

$$x = x_1 + d_{\varphi_n}^{\alpha} x_2 + x_3 d_{\varphi_n}^{\alpha},$$

with d_{φ_n} standing for the density associated to φ_n and $\alpha = 1/p - 1/2$. Note that $\mathcal{K}_{rc_p}^p(\psi_n)$ coincides algebraically with $L_p(q_n \mathcal{M} q_n)$. There exists a close relation between $\mathcal{K}_{rc_p}^p(\psi_n)$ and conditional L_p spaces. Indeed, let us consider the conditional expectation $\mathsf{E}_{\varphi_n} : \mathrm{M}_m(q_n \mathcal{M} q_n) \to \mathrm{M}_m$ given by

$$\mathsf{E}_{\varphi_n}\big((x_{ij})\big) = \big(\varphi_n(x_{ij})\big) = \Big(\frac{\psi_n(x_{ij})}{\mathrm{k}_n}\Big).$$

Lemma 4.15. *We have isometries*

$$\begin{aligned} S_p^m\big(L_2^{r_p}(q_n \mathcal{M} q_n)\big) &= m^{\frac{1}{p}} L_p^r\big(\mathrm{M}_m(q_n \mathcal{M} q_n), \mathsf{E}_{\varphi_n}\big), \\ S_p^m\big(L_2^{c_p}(q_n \mathcal{M} q_n)\big) &= m^{\frac{1}{p}} L_p^c\big(\mathrm{M}_m(q_n \mathcal{M} q_n), \mathsf{E}_{\varphi_n}\big). \end{aligned}$$

Moreover, these isometries have the form

$$\begin{aligned} \big\|d_{\varphi_n}^{\frac{1}{2}} a\big\|_{S_p^m(L_2^{r_p}(q_n \mathcal{M} q_n))} &= m^{\frac{1}{p}} \big\|d_{\varphi_n}^{\frac{1}{p}} a\big\|_{L_p^r(\mathrm{M}_m(q_n \mathcal{M} q_n), \mathsf{E}_{\varphi_n})}, \\ \big\|a d_{\varphi_n}^{\frac{1}{2}}\big\|_{S_p^m(L_2^{c_p}(q_n \mathcal{M} q_n))} &= m^{\frac{1}{p}} \big\|a d_{\varphi_n}^{\frac{1}{p}}\big\|_{L_p^c(\mathrm{M}_m(q_n \mathcal{M} q_n), \mathsf{E}_{\varphi_n})}. \end{aligned}$$

In particular, using the relation $d_{\varphi_n}^{\frac{1}{p}} = d_{\varphi_n}^{\frac{1}{2}} d_{\varphi_n}^{\alpha}$, we conclude

$$\begin{aligned} &\|x\|_{S_p^m(\mathcal{K}_{rc_p}^p(\psi_n))} \\ &= \inf_{x = x_p + x_r + x_c} \Big\{ \mathrm{k}_n^{\frac{1}{p}} \|x_p\|_p + \mathrm{k}_n^{\frac{1}{2}} \big\|\mathsf{E}_{\varphi_n}(x_r x_r^*)^{\frac{1}{2}}\big\|_{S_p^m} + \mathrm{k}_n^{\frac{1}{2}} \big\|\mathsf{E}_{\varphi_n}(x_c^* x_c)^{\frac{1}{2}}\big\|_{S_p^m} \Big\}. \end{aligned}$$

Proof. We have

$$\begin{aligned} \big\|d_{\varphi_n}^{\frac{1}{2}} a\big\|_{S_p^m(L_2^{r_p}(q_n \mathcal{M} q_n))} &= \Big\| \mathrm{tr}_{\mathcal{M}}\big(d_{\varphi_n}^{\frac{1}{2}} a a^* d_{\varphi_n}^{\frac{1}{2}}\big)^{\frac{1}{2}} \Big\|_{S_p^m} \\ &= \Big\| \big(id_{\mathrm{M}_m} \otimes \varphi_n\big)(a a^*)^{\frac{1}{2}} \Big\|_{S_p^m}. \end{aligned}$$

Then, normalizing the trace on M_m and recalling that

$$\Big\| \big(id_{\mathrm{M}_m} \otimes \varphi_n\big)(a a^*)^{\frac{1}{2}} \Big\|_{S_p^m} = m^{\frac{1}{p}} \Big\| \mathsf{E}_{\varphi_n}\big(d_{\varphi_n}^{\frac{1}{p}} a a^* d_{\varphi_n}^{\frac{1}{p}}\big)^{\frac{1}{2}} \Big\|_{L_p(\mathrm{M}_m(q_n \mathcal{M} q_n))}$$

when regarding the conditional expectation as a mapping

$$\mathsf{E}_{\varphi_n} : L_p\big(\mathrm{M}_m(q_n \mathcal{M} q_n)\big) \to L_p(\mathrm{M}_m),$$

we deduce the assertion. The column case is proved in the same way. $\square$

Proposition 4.16. *Let $\mathcal{R}$ be the hyperfinite* III_1 *factor and ϕ the quasi-free state on $\mathcal{R}$ considered above. Let us consider the space $\mathcal{K}^p_{rc_p}(\phi \otimes \psi_n)$, defined as we did above. Then, there exists a completely isomorphic embedding*

$$\rho_n : \mathcal{K}_{p,2}(\psi_n) \to \mathcal{K}^p_{rc_p}(\phi \otimes \psi_n).$$

Moreover, the relevant constants in ρ_n are independent of n.

Proof. We have

$$\mathcal{K}^p_{rc_p}(\phi \otimes \psi_n) = \mathrm{k}_n^{\frac{1}{p}} L_p(\mathcal{R} \bar{\otimes} q_n \mathcal{M} q_n) + \mathrm{k}_n^{\frac{1}{2}} L_2^{r_p}(\mathcal{R} \bar{\otimes} q_n \mathcal{M} q_n) + \mathrm{k}_n^{\frac{1}{2}} L_2^{c_p}(\mathcal{R} \bar{\otimes} q_n \mathcal{M} q_n).$$

The embedding is given by $\rho_n(x) = \gamma_p \otimes x$, with γ_p the element of $L_p(\mathcal{R})$ introduced before Proposition 4.12. Indeed, taking $\mathsf{E}_{\phi \otimes \varphi_n} : \mathrm{M}_m(\mathcal{R} \bar{\otimes} q_n \mathcal{M} q_n) \to \mathrm{M}_m$ and $x \in S_p^m(L_p(q_n \mathcal{M} q_n))$, we may argue as above and obtain

$$\begin{aligned} & \big\| \gamma_p \otimes x \big\|_{S_p^m(\mathcal{K}^p_{rc_p}(\phi \otimes \psi_n))} \\ & \quad = \inf_{\gamma_p \otimes x = x_p + x_r + x_c} \Big\{ \mathrm{k}_n^{\frac{1}{p}} \|x_p\|_p + \mathrm{k}_n^{\frac{1}{2}} \big\| \mathsf{E}_{\phi \otimes \varphi_n}(x_r x_r^*)^{\frac{1}{2}} \big\|_{S_p^m} + \mathrm{k}_n^{\frac{1}{2}} \big\| \mathsf{E}_{\phi \otimes \varphi_n}(x_c^* x_c)^{\frac{1}{2}} \big\|_{S_p^m} \Big\}. \end{aligned}$$

Therefore, Lemma 4.9 and Remark 4.10 give

$$\|\gamma_p \otimes x\|_{S_p^m(\mathcal{K}^p_{rc_p}(\phi \otimes \psi_n))} \sim \Big\| \sum_{k=1}^{\mathrm{k}_n} \pi^j_{ind}(\gamma_p \otimes x) \Big\|_p \sim \Big\| \sum_{j=1}^{\mathrm{k}_n} \pi^j_{ind}(\gamma_p) \otimes \pi^j_{free}(x, -x) \Big\|_p.$$

Hence, the assertion follows as in Proposition 4.12. The proof is complete. $\square$

Let $\mathcal{M}$ be a von Neumann algebra equipped with a *n.s.s.f.* weight ψ and let us write $(\psi_n, q_n)_{n \ge 1}$ for the associated sequence of q_n-supported weights. Then we define the following direct limit

$$\mathcal{K}^p_{rc_p}(\psi) = \overline{\bigcup_{n \ge 1} \mathcal{K}^p_{rc_p}(\psi_n)}.$$

We are interested in a cb-embedding $\mathcal{K}^p_{rc_p}(\psi) \to L_p(\mathcal{A})$ preserving hyperfiniteness. In the construction, we shall use a noncommutative Poisson random measure. Let us briefly review the main properties of this notion from [14] before stating our result. Let $\mathcal{M}^f_{sa}$ stand for the subspace of self-adjoint elements in $\mathcal{M}$ which are ψ-finitely supported. Let $\mathcal{M}_\pi$ denote the projection lattice of $\mathcal{M}$. We write $e \perp f$ for orthogonal projections. A *noncommutative Poisson random measure* is a map $\lambda : (\mathcal{M}, \psi) \to L_1(\mathcal{A}, \Phi_\psi)$, where $(\mathcal{A}, \Phi_\psi)$ is a noncommutative probability space and the following conditions hold

- **(i)** $\lambda : \mathcal{M}^f_{sa} \to L_1(\mathcal{A})$ is linear.
- **(ii)** $\Phi_\psi(e^{i\lambda(x)}) = \exp(\psi(e^{ix} - 1))$ for $x \in \mathcal{M}^f_{sa}$.
- **(iii)** If $e, f \in \mathcal{M}_\pi$ and $e \perp f$, $\lambda(e\mathcal{M}e)''$ and $\lambda(f\mathcal{M}f)''$ are strongly independent.

These properties are not yet enough to characterize λ, see below. Let us recall that two von Neumann subalgebras $\mathcal{A}_1, \mathcal{A}_2$ of $\mathcal{A}$ are called strongly independent if $a_1 a_2 = a_2 a_1$ and $\Phi_\psi(a_1 a_2) = \Phi_\psi(a_1)\Phi_\psi(a_2)$ for any pair (a_1, a_2) in $\mathcal{A}_1 \times \mathcal{A}_2$. The

construction of λ follows by a direct limit argument. Indeed, let us show how to produce $\lambda : (q_n\mathcal{M}q_n, \psi_n) \to L_1(\mathcal{A}_n, \Phi_{\psi_n})$. We define

$$\mathcal{A}_n = M_s(q_n\mathcal{M}q_n) = \prod_{k=0}^{\infty} \otimes_s^k q_n\mathcal{M}q_n,$$

where $\otimes_s^k q_n\mathcal{M}q_n$ denotes the subspace of symmetric tensors in the k-fold tensor product $(q_n\mathcal{M}q_n)_{\otimes k}$. In other words, if $\mathcal{S}_k$ is the symmetric group of permutations of k elements, the space $\otimes_s^k q_n\mathcal{M}q_n$ is the range of the conditional expectation

$$\mathcal{E}_k(x_1 \otimes x_2 \otimes \cdots \otimes x_k) = \frac{1}{k!} \sum_{\pi \in \mathcal{S}_k} x_{\pi(1)} \otimes x_{\pi(2)} \otimes \cdots \otimes x_{\pi(k)}.$$

Then we set

$$\lambda(x) = (\lambda_k(x))_{k\geq 0} \in M_s(q_n\mathcal{M}q_n) \quad \text{with} \quad \lambda_k(x) = \sum_{j=1}^{k} \pi_{ind}^j(x),$$

and properties (i), (ii) and (iii) hold when working with the state

$$\Phi_{\psi_n}\big((z_k)_{k\geq 1}\big) = \sum_{k=0}^{\infty} \frac{\exp(-\psi_n(1))}{k!} \underbrace{\psi_n \otimes \cdots \otimes \psi_n}_{k \text{ times}}(z_k).$$

In the following, it will be important to know the moments with respect to this state. Given $m \geq 1$, $\Pi(m)$ will be the set of partitions of $\{1, 2, \ldots, m\}$. On the other hand, given an ordered family $(x_\alpha)_{\alpha\in\Lambda}$ in $\mathcal{M}$, we shall write

$$\overrightarrow{\prod_{\alpha\in\Lambda}} x_\alpha$$

for the directed product of the x_α's. Then, the moments are given by the formula

$$\textbf{(iv)} \ \ \Phi_{\psi_n}\big(\lambda(x_1)\lambda(x_2)\cdots\lambda(x_m)\big) = \sum_{\substack{\sigma\in\Pi(m) \\ \sigma=\{\sigma_1,\ldots,\sigma_r\}}} \prod_{k=1}^{r} \psi_n\Big(\overrightarrow{\prod_{j\in\sigma_k}} x_j\Big).$$

Now we can say that properties (i)-(iv) determine the Poisson random measure λ for any given *n.s.s.f.* weight ψ in $\mathcal{M}$. According to a uniqueness result from [13] which provides a noncommutative form of the Hamburger moment problem, it turns out that there exists a state preserving embedding of $M_s(q_{n_1}\mathcal{M}q_{n_1})$ into $M_s(q_{n_2}\mathcal{M}q_{n_2})$ for $n_1 \leq n_2$ and such that the map $\lambda = \lambda_{n_1}$ constructed for q_{n_1} may be obtained as a restriction of λ_{n_2}. This allows to take direct limits. More precisely, let us define $M_s(\mathcal{M})$ to be the ultra-weak closure of the direct limit of the $M_s(q_n\mathcal{M}q_n)$'s. Then, there exists a *n.f.* state Φ_ψ on $M_s(\mathcal{M})$ and a map λ which assigns to every self-adjoint operator x (with $\mathrm{supp}(x) \leq e$ for some ψ-finite projection e in $\mathcal{M}$) a self-adjoint unbounded operator $\lambda(x)$ affiliated to $M_s(\mathcal{M})$ and such that

$$\Phi_\psi(e^{i\lambda(x)}) = \exp(\psi(e^{ix} - 1)).$$

Theorem 4.17. *Let $1 \leq p \leq 2$ and ψ be a n.s.s.f. weight on $\mathcal{M}$. Then there exists a von Neumann algebra $\mathcal{A}$, which is hyperfinite when $\mathcal{M}$ is hyperfinite, and a completely isomorphic embedding*

$$\mathcal{K}_{rc_p}^p(\psi) \to L_p(\mathcal{A}).$$

Proof. Let us set the s-fold tensor product

$$\mathcal{B}_{n,s} = \Big(L_\infty[0,1] \bar{\otimes} \big[q_n \mathcal{M} q_n \oplus q_n \mathcal{M} q_n \big] \Big)_{\otimes s}.$$

Given $s \ge \mathrm{k}_n$, we define the mapping $\Lambda_{n,s} : \mathcal{K}_{rc_p}^p(\psi_n) \to L_p(\mathcal{B}_{n,s})$ by

$$\begin{aligned} \Lambda_{n,s}\big(d_{\varphi_n}^{\frac{1}{2p}} x d_{\varphi_n}^{\frac{1}{2p}}\big) &= \sum_{j=1}^s \pi_{ind}^j \Big(1_{[0,\mathrm{k}_n/s]} \otimes d_{\varphi_n}^{\frac{1}{2p}} \, (x,-x) \, d_{\varphi_n}^{\frac{1}{2p}} \Big) \\ &= \sum_{j=1}^s d_{n,s}^{\frac{1}{2p}} \, \pi_{ind}^j \big(1_{[0,\mathrm{k}_n/s]} \otimes (x,-x) \big) \, d_{n,s}^{\frac{1}{2p}}, \end{aligned}$$

where $d_{n,s}$ is the density associated to the s-fold tensor product state

$$\phi_{n,s} = \Big[\int_0^1 \cdot \, dt \otimes \frac{1}{2}(\varphi_n \oplus \varphi_n) \Big]_{\otimes s} = \underbrace{\phi_n \otimes \phi_n \otimes \cdots \otimes \phi_n}_{s \text{ times}}.$$

If we tensor $\Lambda_{n,s}$ with the identity map on M_m, the resulting mapping gives a sum of symmetrically independent mean-zero random variables over M_m. Therefore, taking $x \in S_p^m(\mathcal{K}_{rc_p}^p(\psi_n))$ and applying Lemma 4.9/Remark 4.10

$$\big\| \Lambda_{n,s}\big(d_{\varphi_n}^{\frac{1}{2p}} x d_{\varphi_n}^{\frac{1}{2p}}\big) \big\|_p \ \sim \ \inf \Big\{ s^{\frac{1}{p}} \|a\|_p + s^{\frac{1}{2}} \big\| \mathsf{E}_{\phi_n}(bb^*)^{\frac{1}{2}} \big\|_{S_p^m} + s^{\frac{1}{2}} \big\| \mathsf{E}_{\phi_n}(c^*c)^{\frac{1}{2}} \big\|_{S_p^m} \Big\},$$

where E_{ϕ_n} denotes the conditional expectation

$$\mathsf{E}_{\phi_n} : \mathrm{M}_m\Big(L_\infty[0,1] \bar{\otimes} \big[q_n \mathcal{M} q_n \oplus q_n \mathcal{M} q_n \big] \Big) \to \mathrm{M}_m$$

and the infimum runs over all possible decompositions

$$1_{[0,\mathrm{k}_n/s]} \otimes d_{\varphi_n}^{\frac{1}{2p}} \, (x,-x) \, d_{\varphi_n}^{\frac{1}{2p}} = a + b + c. \tag{4.4}$$

Multiplying at both sizes by $1_{[0,\mathrm{k}_n/s]} \otimes 1$, we obtain a new decomposition which vanishes over $(\mathrm{k}_n/s, 1]$. Thus, since this clearly improves the infimum above, we may assume this property in all decompositions considered. Moreover, we claim that we can also restrict the infimum above to those decompositions $a+b+c$ which are constant on $[0, \mathrm{k}_n/s]$. Indeed, given any decomposition of the form (4.4) we take averages at both sizes and produce another decomposition $a_0 + b_0 + c_0$ given by the relations

$$\big(a_0, b_0, c_0\big) = 1_{[0,\mathrm{k}_n/s]} \otimes \frac{s}{\mathrm{k}_n} \int_0^{\frac{\mathrm{k}_n}{s}} \big(a(t), b(t), c(t)\big) \, dt.$$

Then, our claim is a consequence of the inequalities

$$\begin{aligned} \|a_0\|_p &\le \|a\|_p, \\ \big\| \mathsf{E}_{\phi_n}(b_0 b_0^*)^{\frac{1}{2}} \big\|_{S_p^m} &\le \big\| \mathsf{E}_{\phi_n}(bb^*)^{\frac{1}{2}} \big\|_{S_p^m}, \\ \big\| \mathsf{E}_{\phi_n}(c_0^* c_0)^{\frac{1}{2}} \big\|_{S_p^m} &\le \big\| \mathsf{E}_{\phi_n}(c^*c)^{\frac{1}{2}} \big\|_{S_p^m}. \end{aligned}$$

The fist one is justified by means of the inequality

$$\lambda^{\frac{1}{p}} \Big\| \frac{1}{\lambda} \int_0^\lambda a(t) \, dt \Big\|_{L_p(\mathcal{M})} \le \|a\|_{L_p([0,\lambda] \bar{\otimes} \mathcal{M})},$$

which follows easily by complex interpolation. The two other inequalities arise as a consequence of Kadison's inequality $\mathsf{E}(x)\mathsf{E}(x^*) \le \mathsf{E}(xx^*)$ applied to the conditional expectation

$$\mathsf{E} = \frac{1}{\lambda} \int_0^\lambda \cdot \, dt.$$

Our considerations allow us to assume

$$\big(a, b, c\big) = 1_{[0, \mathrm{k}_n/s]} \otimes \big(x_p, x_r, x_c\big)$$

for some $x_p, x_r, x_c \in S_p^m(L_p(q_n \mathcal{M} q_n))$. This gives rise to

$$\big\| \Lambda_{n,s} \big(d_{\varphi_n}^{\frac{1}{2p}} x d_{\varphi_n}^{\frac{1}{2p}} \big) \big\|_p \sim \inf \Big\{ \mathrm{k}_n^{\frac{1}{p}} \|x_p\|_p + \mathrm{k}_n^{\frac{1}{2}} \big\| \mathsf{E}_{\varphi_n}(x_r x_r^*)^{\frac{1}{2}} \big\|_{S_p^m} + \mathrm{k}_n^{\frac{1}{2}} \big\| \mathsf{E}_{\varphi_n}(x_c^* x_c)^{\frac{1}{2}} \big\|_{S_p^m} \Big\},$$

where the infimum runs over all possible decompositions

$$d_{\varphi_n}^{\frac{1}{2p}} x d_{\varphi_n}^{\frac{1}{2p}} = x_p + x_r + x_c.$$

This shows that $\Lambda_{n,s} : \mathcal{K}_{rc_p}^p(\psi_n) \to L_p(\mathcal{B}_{n,s})$ is a completely isomorphic embedding with constants independent of n or s. We are not ready yet to take direct limits. Before that, we use the algebraic central limit theorem to identify the moments in the limit as $s \to \infty$. To calculate the joint moments we set

$$\zeta_n = \frac{1}{2}(\varphi_n \oplus \varphi_n) \quad \text{and} \quad \zeta_{n,s} = \underbrace{\zeta_n \otimes \zeta_n \otimes \cdots \otimes \zeta_n}_{s \text{ times}}$$

and recall that the map $\Lambda_{n,s}$ corresponds to

$$u_{n,s}(x) = \sum_{j=1}^s \pi_{ind}^j \Big(1_{[0, \mathrm{k}_n/s]} \otimes (x, -x) \Big).$$

Then, the joint moments are given by

$$\begin{aligned} & \phi_{n,s}\big(u_{n,s}(x_1) \cdots u_{n,s}(x_m)\big) \\ & = \sum_{j_1, j_2, \ldots, j_m = 1}^s \int_{[0,1]^s} \prod_{i=1}^m \pi_{ind}^{j_i}(1_{[0,\mathrm{k}_n/s]})(t) \, dt \ \zeta_{n,s} \Big(\prod_{1 \le i \le m}^{\longrightarrow} \pi_{ind}^{j_i}(x_i, -x_i) \Big) \\ & = \sum_{\substack{\sigma \in \Pi(m) \\ \sigma = \{\sigma_1, \ldots, \sigma_r\}}} \ \sum_{(j_1, \ldots, j_m) \sim \sigma} \Big(\frac{\mathrm{k}_n}{s} \Big)^r \prod_{k=1}^r \zeta_n \Big[\Big(\prod_{i \in \sigma_k}^{\longrightarrow} x_i, (-1)^{|\sigma_k|} \prod_{i \in \sigma_k}^{\longrightarrow} x_i \Big) \Big], \end{aligned}$$

where $|\sigma_k|$ denotes the cardinality of σ_k and we write $(j_1, \ldots, j_m) \sim \sigma$ when $j_a = j_b$ if and only if there exits $1 \le k \le r$ such that $j_a, j_b \in \sigma_k$. Therefore, recalling that $\zeta_n = \frac{1}{2}(\varphi_n \oplus \varphi_n)$, the only partitions which contribute to the sum above are the even partitions satisfying $|\sigma_k| \in 2\mathbb{N}$ for $1 \le k \le r$. Let us write $\Pi_e(m)$ for the set of even partitions. Then, using $\psi_n = \mathrm{k}_n \varphi_n$ we deduce

$$\begin{aligned} \phi_{n,s}\big(u_{n,s}(x_1) \cdots u_{n,s}(x_m)\big) & = \sum_{\substack{\sigma \in \Pi_e(m) \\ \sigma = \{\sigma_1, \ldots, \sigma_r\}}} \frac{|\{(j_1, \ldots, j_m) \sim \sigma\}|}{s^r} \prod_{k=1}^r \psi_n \Big(\prod_{i \in \sigma_k}^{\longrightarrow} x_i \Big) \\ & = \sum_{\substack{\sigma \in \Pi_e(m) \\ \sigma = \{\sigma_1, \ldots, \sigma_r\}}} \frac{s!}{s^r (s-r)!} \prod_{k=1}^r \psi_n \Big(\prod_{i \in \sigma_k}^{\longrightarrow} x_i \Big). \end{aligned}$$

Therefore, taking limits

$$\lim_{s\to\infty} \phi_{n,s}\big(u_{n,s}(x_1)\cdots u_{n,s}(x_m)\big) = \sum_{\substack{\sigma\in\Pi_e(m)\\ \sigma=\{\sigma_1,\ldots,\sigma_r\}}} \prod_{k=1}^{r} \psi_n\Big(\overrightarrow{\prod_{i\in\sigma_k}} x_i\Big).$$

These moments coincide with the moments of the Poisson random process

$$\lambda : (q_n\mathcal{M}q_n, \psi_n) \to L_1(M_s(q_n\mathcal{M}q_n), \Phi_{\psi_n}).$$

Hence, the noncommutative version of the Hamburger moment problem from [13] provides a state preserving homomorphism between the von Neumann algebra which generate the operators

$$\Big\{e^{iu_{n,s}(x)} \ \big|\ x\in q_n\mathcal{M}q_n,\ s\geq 1\Big\}$$

and the von Neumann subalgebra of $M_s(q_n\mathcal{M}q_n)$ generated by

$$\Big\{e^{i\lambda(x)} \ \big|\ x\in (q_n\mathcal{M}q_n)^f_{sa}\Big\}.$$

In particular, taking $\mathcal{A}_n = M_s(q_n\mathcal{M}q_n)$

$$\begin{aligned}&\big\|d_{\psi_n}^{\frac{1}{2p}} x d_{\psi_n}^{\frac{1}{2p}}\big\|_{S_p^m(\mathcal{K}^p_{rc_p}(\psi_n))}\\ &\quad\sim \lim_{s\to\infty} \big\|\Lambda_{n,s}\big(d_{\psi_n}^{\frac{1}{2p}} x d_{\psi_n}^{\frac{1}{2p}}\big)\big\|_{S_p^m(L_p(\mathcal{B}_{n,s}))} = \big\|d_{\Phi_{\psi_n}}^{\frac{1}{2p}} \lambda(x) d_{\Phi_{\psi_n}}^{\frac{1}{2p}}\big\|_{S_p^m(L_p(\mathcal{A}_n))}.\end{aligned}$$

Now, we use from [14] that

$$\big(M_s(\mathcal{M}), \Phi_\psi\big) = \overline{\bigcup_{n\geq 1} \big(M_s(q_n\mathcal{M}q_n), \Phi_{\psi_n}\big)}$$

exists. Therefore, the map

$$\Lambda\big(d_\psi^{\frac{1}{2p}} x d_\psi^{\frac{1}{2p}}\big) = d_{\Phi_\psi}^{\frac{1}{2p}} \lambda(x) d_{\Phi_\psi}^{\frac{1}{2p}}$$

extends to a complete embedding

$$\mathcal{K}^p_{rc_p}(\psi) = \lim\nolimits_n \mathcal{K}^p_{rc_p}(\psi_n) \to L_p(M_s(\mathcal{M})).$$

Moreover, if $\mathcal{M}$ is hyperfinite so is $M_s(q_n\mathcal{M}q_n)$ and hence the limit $M_s(\mathcal{M})$. $\square$

Corollary 4.18. *Let $1\leq p\leq 2$ and ψ be a n.s.s.f. weight on $\mathcal{M}$. Then there exists a von Neumann algebra $\mathcal{A}$, which is hyperfinite when $\mathcal{M}$ is hyperfinite, and a completely isomorphic embedding*

$$\mathcal{K}_{p,2}(\psi) \to L_p(\mathcal{A}).$$

Proof. Let us set

$$\mathcal{RB}_{n,s} = \Big(L_\infty[0,1]\bar{\otimes}\big[(\mathcal{R}\bar{\otimes}q_n\mathcal{M}q_n)\oplus(\mathcal{R}\bar{\otimes}q_n\mathcal{M}q_n)\big]\Big)_{\otimes s}.$$

By Proposition 4.16 and Theorem 4.17, the map

$$\Lambda_{n,s}\circ\rho_n : \mathcal{K}_{p,2}(\psi_n) \to \mathcal{K}^p_{rc_p}(\phi\otimes\psi_n) \to L_p(\mathcal{RB}_{n,s})$$

provides a complete isomorphism with constant independent of n and s. Using the algebraic central limit theorem to take limits in s and the noncommutative version of the Hamburger moment problem one more time, we obtain a complete embedding

$$\Lambda_n\circ\rho_n : \mathcal{K}_{p,2}(\psi_n) \to \mathcal{K}^p_{rc_p}(\phi\otimes\psi_n) \to L_p\big(M_s(\mathcal{R}\bar{\otimes}q_n\mathcal{M}q_n)\big).$$

Taking direct limits we obtain a cb-embedding which preserves hyperfiniteness. □

Corollary 4.19. *S_q cb-embeds into $L_p(\mathcal{A})$ for some hyperfinite factor $\mathcal{A}$.*

Proof. According to the complete isometry $S_q = C_q \otimes_h R_q$ and Remark 1.3, it suffices to embed the first term $\mathcal{Z}_1 \otimes \mathcal{Z}_2$ on the right of (4.1) into $L_p(\mathcal{A})$ for some hyperfinite von Neumann algebra $\mathcal{A}$. However, following Part II of the proof of Theorem 4.4, we know that $\mathcal{Z}_1 \otimes \mathcal{Z}_2$ embeds completely isomorphically into $\mathcal{K}_{p,2}(\psi)$, where ψ denotes some *n.s.s.f.* weight on $\mathcal{B}(\ell_2)$. Therefore, the assertion follows from Corollary 4.18. This completes the proof. □

Remark 4.20. Theorem 4.4 easily generalizes to the context of Corollary 4.19. More precisely, given operator spaces $\mathrm{X}_1 \in \mathcal{QS}(C_p \oplus_2 \mathrm{OH})$ and $\mathrm{X}_2 \in \mathcal{QS}(R_p \oplus_2 \mathrm{OH})$ and combining the techniques applied so far, it is rather easy to find a hyperfinite type III factor $\mathcal{A}$ and a completely isomorphic embedding

$$\mathrm{X}_1 \otimes_h \mathrm{X}_2 \to L_p(\mathcal{A}).$$

Remark 4.21. In contrast with Corollary 4.8, where free products are used, the complete embedding of S_q into L_p given in Corollary 4.19 provides estimates on the dimension of $\mathcal{A}$ in the cb-embedding

$$S_q^m \to L_p(\mathcal{A}).$$

Indeed, a quick look at our construction shows that

$$S_q^m = C_q^m \otimes_h R_q^m \to \mathcal{K}_{p,2}(\psi_n) \to L_p\big(\mathcal{A}_{ind}^n; \mathrm{OH}_{\mathrm{k}_n}\big) = L_p\big(\mathrm{M}_n^{\otimes \mathrm{k}_n}; \mathrm{OH}_{\mathrm{k}_n}\big),$$

with $n \sim m \log m$, see [13] for this last assertion. This chain essentially follows from Remark 1.3, Lemma 4.7 and Proposition 4.12. On the other hand, given any parameter $\gamma > 0$ and according once more to to [13], we know that $\mathrm{OH}_{\mathrm{k}_n}$ embeds completely isomorphically into $S_p^{\mathrm{w}_n}$ for $\mathrm{w}_n = \mathrm{k}_n^{\gamma \mathrm{k}_n}$ with absolute constants depending only on γ and that $\mathrm{k}_n \sim n^{\alpha_p}$. Combining the embeddings mentioned so far, we have found a complete embedding

$$S_q^m \to S_p^M \quad \text{with} \quad M \sim m^{\beta_p m^{\alpha_p}}.$$

4.3. **Embedding for general von Neumann algebras.** We conclude this paper with the proof of our main embedding result in full generality. We shall need to extend our definition (2.8) to the case $1 \le q \le 2$. This is easily done as follows

$$\begin{array}{rcl} L_q^r(\mathcal{M}) & = & \big[L_1(\mathcal{M}), L_2^r(\mathcal{M})\big]_{\frac{2}{q'}}, \\ L_q^c(\mathcal{M}) & = & \big[L_1(\mathcal{M}), L_2^c(\mathcal{M})\big]_{\frac{2}{q'}}. \end{array} \tag{4.5}$$

The space L_q is given by complex interpolation. Therefore we will encode complex interpolation in a suitable graph. This follows Pisier's approach [36] to the main result in [11]. Indeed, consider a fixed $0 < \theta < 1$ and let $\mu_\theta = (1-\theta)\mu_0 + \theta\mu_1$ be the harmonic measure on the boundary of the strip $\mathcal{S}$ associated to the point $z = \theta$, as defined at the beginning of Section 1. We consider the space

$$\mathcal{H}_2 = \Big\{ \big(f_{|\partial_0}, f_{|\partial_1}\big) \;\big|\; f : \mathcal{S} \to \mathbb{C} \ \text{ analytic} \Big\} \subset L_2(\partial_0) \oplus L_2(\partial_1).$$

We need operator-valued versions of this space given by subspaces

$$\mathcal{H}_{2p',2}^r(\mathcal{M},\theta) \subset \Big(L_2^{c_{p'}}(\partial_0) \otimes_h L_{2p'}^r(\mathcal{M})\Big) \oplus \Big(L_2^{oh}(\partial_1) \otimes_h L_4^r(\mathcal{M})\Big) = \mathcal{O}_{p',0}^r \oplus \mathcal{O}_{p',1}^r,$$

$$\mathcal{H}_{2p',2}^c(\mathcal{M},\theta) \subset \Big(L_{2p'}^c(\mathcal{M}) \otimes_h L_2^{r_{p'}}(\partial_0)\Big) \oplus \Big(L_4^c(\mathcal{M}) \otimes_h L_2^{oh}(\partial_1)\Big) = \mathcal{O}_{p',0}^c \oplus \mathcal{O}_{p',1}^c.$$

More precisely, if $\mathcal{M}$ comes equipped with a *n.s.s.f.* weight ψ and d_ψ denotes the associated density, $\mathcal{H}^r_{2p',2}(\mathcal{M},\theta)$ is the subspace of all pairs (f_0,f_1) of functions in $\mathcal{O}^r_{p',0}\oplus\mathcal{O}^r_{p',1}$ such that for every scalar-valued analytic function $g:\mathcal{S}\to\mathbb{C}$ (extended non-tangentially to the boundary) with $g(\theta)=0$, we have

$$(1-\theta)\int_{\partial_0} g(z)\, d_\psi^{\frac14-\frac{1}{2p'}} f_0(z)\, d\mu_0(z)+\theta\int_{\partial_1} g(z)f_1(z)\, d\mu_1(z)=0.$$

Similarly, the condition on $\mathcal{H}^c_{2p',2}(\mathcal{M},\theta)$ is

$$(1-\theta)\int_{\partial_0} g(z)f_0(z)\, d_\psi^{\frac14-\frac{1}{2p'}}\, d\mu_0(z)+\theta\int_{\partial_1} g(z)f_1(z)\, d\mu_1(z)=0.$$

We shall also need to consider the subspaces

$$\begin{aligned}\mathcal{H}_{r,0} &= \Big\{(f_0,f_1)\in\mathcal{H}^r_{2p',2}(\mathcal{M},\theta)\,\big|\,(1-\theta)\int_{\partial_0} d_\psi^{\frac14-\frac{1}{2p'}} f_0 d\mu_0+\theta\int_{\partial_1} f_1 d\mu_1=0\Big\},\\ \mathcal{H}_{c,0} &= \Big\{(f_0,f_1)\in\mathcal{H}^c_{2p',2}(\mathcal{M},\theta)\,\big|\,(1-\theta)\int_{\partial_0} f_0 d_\psi^{\frac14-\frac{1}{2p'}} d\mu_0+\theta\int_{\partial_1} f_1 d\mu_1=0\Big\}.\end{aligned}$$

Remark 4.22. In order to make all the forthcoming duality arguments work, we need to introduce a slight modification of these spaces for $p=1$. Indeed, in that case the spaces defined above must be regarded as subspaces of

$$\begin{aligned}\mathcal{H}^r_{\infty,2}(\mathcal{M},\theta) &\subset \Big(L_2^c(\partial_0)\bar\otimes\mathcal{M}\Big)\oplus\Big(L_2^{oh}(\partial_1)\otimes_h L_4^r(\mathcal{M})\Big),\\ \mathcal{H}^c_{\infty,2}(\mathcal{M},\theta) &\subset \Big(\mathcal{M}\bar\otimes L_2^r(\partial_0)\Big)\oplus\Big(L_4^c(\mathcal{M})\otimes_h L_2^{oh}(\partial_1)\Big).\end{aligned}$$

The von Neumann algebra tensor product used above is the weak closure of the minimal tensor product, which in this particular case coincides with the Haagerup tensor product since we have either a column space on the left or a row space on the right. In particular, we just take the closure in the weak operator topology.

Lemma 4.23. *Let $\mathcal{M}$ be a finite von Neumann algebra equipped with a n.f. state φ and let d_φ be the associated density. If $2\le q'<p'$ and $\frac{1}{2q'}=\frac{1-\theta}{2p'}+\frac{\theta}{4}$, we have complete contractions*

$$u_r: d_\varphi^{\frac{1}{2q'}}x\in L^r_{2q'}(\mathcal{M})\mapsto\Big(1\otimes d_\varphi^{\frac{1}{2p'}}x,1\otimes d_\varphi^{\frac14}x\Big)+\mathcal{H}_{r,0}\in\mathcal{H}^r_{2p',2}(\mathcal{M},\theta)/\mathcal{H}_{r,0},$$

$$u_c: x d_\varphi^{\frac{1}{2q'}}\in L^c_{2q'}(\mathcal{M})\mapsto\Big(x d_\varphi^{\frac{1}{2p'}}\otimes 1,x d_\varphi^{\frac14}\otimes 1\Big)+\mathcal{H}_{c,0}\in\mathcal{H}^c_{2p',2}(\mathcal{M},\theta)/\mathcal{H}_{c,0}.$$

Proof. By symmetry, it suffices to consider the column case. Let x be an element in $\mathrm{M}_m(L^c_{2q'}(\mathcal{M}))$ of norm less than 1. According to our choice of $0<\theta<1$, we find that

$$\mathrm{M}_m(L^c_{2q'}(\mathcal{M}))=\big[\mathrm{M}_m(L^c_{2p'}(\mathcal{M})),\mathrm{M}_m(L^c_4(\mathcal{M}))\big]_\theta.$$

Thus, there exists $f:\mathcal{S}\to\mathrm{M}_m(\mathcal{M})$ analytic such that $f(\theta)=x$ and

$$\max\Big\{\sup_{z\in\partial_0}\big\|f(z)d_\varphi^{\frac{1}{2p'}}\big\|_{\mathrm{M}_m(L^c_{2p'}(\mathcal{M}))},\sup_{z\in\partial_1}\big\|f(z)d_\varphi^{\frac14}\big\|_{\mathrm{M}_m(L^c_4(\mathcal{M}))}\Big\}\le 1.$$

If $1\le s\le\infty$ and $j\in\{0,1\}$, we claim that

$$\big\|f_{|\partial_j}d_\varphi^{\frac{1}{2s}}\big\|_{\mathrm{M}_m(L^c_{2s}(\mathcal{M})\otimes_h L_2^{r_s}(\partial_j))}\le\sup_{z\in\partial_j}\big\|f(z)d_\varphi^{\frac{1}{2s}}\big\|_{\mathrm{M}_m(L^c_{2s}(\mathcal{M}))}. \tag{4.6}$$

Before proving our claim, let us finish the proof. Taking $f_j = f_{|\partial_j}$, we have

$$\big(f_0 d_\varphi^{\frac{1}{2p'}}, f_1 d_\varphi^{\frac{1}{4}}\big) - \Big(x d_\varphi^{\frac{1}{2p'}} \otimes 1, x d_\varphi^{\frac{1}{4}} \otimes 1\Big) \in \mathcal{H}_{c,0}.$$

Indeed by analyticity, we have

$$(1-\theta)\int_{\partial_0} f_0\, d_\varphi^{\frac{1}{2p'}} d_\varphi^{\frac{1}{4}-\frac{1}{2p'}}\, d\mu_0 + \theta \int_{\partial_1} f_1\, d_\varphi^{\frac{1}{4}}\, d\mu_1 = \int_{\partial\mathcal{S}} f d_\varphi^{\frac{1}{4}} d\mu_\theta = f(\theta) d_\varphi^{\frac{1}{4}} = x d_\varphi^{\frac{1}{4}}.$$

This implies from (4.6) applied to $(s,j) = (p',0)$ and $(s,j) = (2,1)$ that

$$\Big\| u_c\big(x d_\varphi^{\frac{1}{2q'}}\big)\Big\|_{\mathrm{M}_m(\mathcal{H}^c_{2p',2}/\mathcal{H}_{c,0})} \le \Big\|\big(f_0 d_\varphi^{\frac{1}{2p'}}, f_1 d_\varphi^{\frac{1}{4}}\big)\Big\|_{\mathrm{M}_m(\mathcal{H}^c_{2p',2})} \le 1.$$

Hence, it remains to prove our claim (4.6). We must show that the identity map $L_\infty\big(\partial_j; L_{2s}^c(\mathcal{M})\big) \to L_{2s}^c(\mathcal{M}) \otimes_h L_2^{r_s}(\partial_j)$ is a complete contraction. By complex interpolation, we have

$$\begin{aligned}
L_\infty\big(\partial_j; L_{2s}^c(\mathcal{M})\big) &= \big[L_\infty\big(\partial_j;\mathcal{M}\big), L_\infty\big(\partial_j; L_2^c(\mathcal{M})\big)\big]_{\frac{1}{s}}, \\
L_{2s}^c(\mathcal{M}) \otimes_h L_2^{r_s}(\partial_j) &= \big[\mathcal{M} \otimes_h L_2^r(\partial_j), L_2^c(\mathcal{M}) \otimes_h L_2^c(\partial_j)\big]_{\frac{1}{s}} \\
&= \big[\mathcal{M} \otimes_{\min} L_2^r(\partial_j), L_2^c(\mathcal{M}) \otimes_{\min} L_2^c(\partial_j)\big]_{\frac{1}{s}}.
\end{aligned}$$

In other words, we must study the identity mappings

$$\begin{aligned}
\mathcal{M} \otimes_{\min} L_\infty(\partial_j) &\to \mathcal{M} \otimes_{\min} L_2^r(\partial_j), \\
L_2^c(\mathcal{M}) \otimes_{\min} L_\infty(\partial_j) &\to L_2^c(\mathcal{M}) \otimes_{\min} L_2^c(\partial_j).
\end{aligned}$$

However, this automatically reduces to see that we have complete contractions

$$\begin{aligned}
L_\infty(\partial_j) &\to L_2^r(\partial_j), \\
L_\infty(\partial_j) &\to L_2^c(\partial_j).
\end{aligned}$$

Therefore it suffices to observe that

$$\begin{aligned}
\|f\|^2_{\mathrm{M}_m(L_2^r(\partial_j))} = \Big\| \int_{\partial_j} f f^* d\mu_j \Big\|_{\mathrm{M}_m} &\le \mu_j(\partial_j) \sup_{z \in \partial_j} \|f(z)\|^2_{\mathrm{M}_m} = \|f\|^2_{\mathrm{M}_m(L_\infty(\partial_j))}, \\
\|f\|^2_{\mathrm{M}_m(L_2^c(\partial_j))} = \Big\| \int_{\partial_j} f^* f d\mu_j \Big\|_{\mathrm{M}_m} &\le \mu_j(\partial_j) \sup_{z \in \partial_j} \|f(z)\|^2_{\mathrm{M}_m} = \|f\|^2_{\mathrm{M}_m(L_\infty(\partial_j))}.
\end{aligned}$$

This completes the proof for $1 < p \le 2$. In the case $p = 1$, we have overlooked the fact that the definition of $\mathcal{H}^c_{\infty,2}(\mathcal{M},\theta)$ (see Remark 4.22 above) is slightly different. The only consequence of this point is that we also need the inequality

$$\big\|f_{|\partial_0}\big\|_{\mathrm{M}_m(\mathcal{M}\bar{\otimes} L_2^r(\partial_0))} \le \sup_{z\in\partial_0} \|f(z)\|_{\mathrm{M}_m(\mathcal{M})}.$$

However, this is proved once more as above. The proof is complete. □

Lemma 4.23 is closely related to Lemma 1.1 and a similar result holds on the preduals. More precisely, we begin by defining the operator-valued Hardy spaces which arise as subspaces

$$\begin{aligned}
\mathcal{H}^c_{2p,2}(\mathcal{M},\theta) &\subset \Big(L_2^{c_p}(\partial_0) \otimes_h L^c_{\frac{2p}{p+1}}(\mathcal{M})\Big) \oplus \Big(L_2^{oh}(\partial_1) \otimes_h L^c_{\frac{4}{3}}(\mathcal{M})\Big), \\
\mathcal{H}^r_{2p,2}(\mathcal{M},\theta) &\subset \Big(L^r_{\frac{2p}{p+1}}(\mathcal{M}) \otimes_h L_2^{r_p}(\partial_0)\Big) \oplus \Big(L^r_{\frac{4}{3}}(\mathcal{M}) \otimes_h L_2^{oh}(\partial_1)\Big),
\end{aligned}$$

formed by pairs (f_0, f_1) respectively satisfying

$$\begin{aligned}(1-\theta)\int_{\partial_0} g(z)f_0(z)\,d\mu_0(z)+\theta\int_{\partial_1} g(z)d_\varphi^{\frac{p+1}{2p}-\frac{3}{4}}f_1(z)\,d\mu_1(z) &= 0,\\ (1-\theta)\int_{\partial_0} g(z)f_0(z)\,d\mu_0(z)+\theta\int_{\partial_1} g(z)f_1(z)d_\varphi^{\frac{p+1}{2p}-\frac{3}{4}}\,d\mu_1(z) &= 0,\end{aligned}$$

for all scalar-valued analytic function $g:\mathcal{S}\to\mathbb{C}$ (extended non-tangentially to the boundary) with $g(\theta)=0$. The subspaces $\mathcal{H}'_{r,0}$ and $\mathcal{H}'_{c,0}$ are defined accordingly. In other words, we have

$$\begin{aligned}\mathcal{H}'_{c,0} &= \Big\{(f_0,f_1)\in\mathcal{H}^c_{2p,2}(\mathcal{M},\theta)\,\big|\,(1-\theta)\int_{\partial_0} f_0\,d\mu_0+\theta\int_{\partial_1} d_\varphi^{\frac{p+1}{2p}-\frac{3}{4}}f_1\,d\mu_1=0\Big\},\\ \mathcal{H}'_{r,0} &= \Big\{(f_0,f_1)\in\mathcal{H}^r_{2p,2}(\mathcal{M},\theta)\,\big|\,(1-\theta)\int_{\partial_0} f_0\,d\mu_0+\theta\int_{\partial_1} f_1 d_\varphi^{\frac{p+1}{2p}-\frac{3}{4}}\,d\mu_1=0\Big\}.\end{aligned}$$

Lemma 4.24. *Let $\mathcal{M}$ be a finite von Neumann algebra equipped with a n.f. state φ and let d_φ be the associated density. Taking the same values for p,q and θ as above, we have complete contractions*

$$\begin{aligned}w_r &: d_\varphi^{\frac{q+1}{2q}}x\in L^c_{\frac{2q}{q+1}}(\mathcal{M})\mapsto\Big(1\otimes d_\varphi^{\frac{p+1}{2p}}x,1\otimes d_\varphi^{\frac{3}{4}}x\Big)+\mathcal{H}'_{c,0}\in\mathcal{H}^c_{2p,2}(\mathcal{M},\theta)/\mathcal{H}'_{c,0},\\ w_c &: xd_\varphi^{\frac{q+1}{2q}}\in L^r_{\frac{2q}{q+1}}(\mathcal{M})\mapsto\Big(xd_\varphi^{\frac{p+1}{2p}}\otimes 1,xd_\varphi^{\frac{3}{4}}\otimes 1\Big)+\mathcal{H}'_{r,0}\in\mathcal{H}^r_{2p,2}(\mathcal{M},\theta)/\mathcal{H}'_{r,0}.\end{aligned}$$

Proof. If $1\le s\le\infty$ and $\mathcal{M}_m=\mathrm{M}_m(\mathcal{M})$, we have

$$\begin{aligned}S_1^m\big(L^r_{\frac{2s}{s+1}}(\mathcal{M})\big) &= S_1^m\Big(\big[L_1(\mathcal{M}),L_2^r(\mathcal{M})\big]_{\frac{1}{s'}}\Big)\\ &= \Big[S_1^m\big(L_1(\mathcal{M})\big),S_2^m\big(L_2(\mathcal{M})\big)S_2^m\Big]_{\frac{1}{s'}}\\ &= S^m_{\frac{2s}{s+1}}\big(L_{\frac{2s}{s+1}}(\mathcal{M})\big)S^m_{2s'}=L_{\frac{2s}{s+1}}(\mathcal{M}_m)S^m_{2s'}.\end{aligned}\tag{4.7}$$

Indeed, the second identity follows from (2.9) and duality (alternatively, one may argue directly as we did in Lemma 4.15), while the third one follows from Theorem 3.2. Now, let us consider an element of norm less than 1

$$xd_\varphi^{\frac{q+1}{2q}}\in S_1^m\big(L^r_{\frac{2q}{q+1}}(\mathcal{M})\big).$$

The isometry above provides a factorization

$$xd_\varphi^{\frac{q+1}{2q}}=\alpha\beta\gamma\delta\in L_{\frac{4}{3}}(\mathcal{M}_m)L_{\rho_1}(\mathcal{M}_m)S^m_{2p'}S^m_{\rho_2}$$

with $\alpha,\beta,\gamma,\delta$ in the unit balls of their respective spaces with

$$\frac{1}{\rho_1}=\frac{q+1}{2q}-\frac{3}{4}\quad\text{and}\quad\frac{1}{\rho_2}=\frac{1}{2q'}-\frac{1}{2p'}.$$

Moreover, by polar decomposition and approximation, we may assume that β and δ are strictly positive elements. In particular, motivated by the complex interpolation isometry

$$L_{\frac{2q}{q+1}}(\mathcal{M}_m)S^m_{2q'}=\Big[L_{\frac{2p}{p+1}}(\mathcal{M}_m)S^m_{2p'},L_{\frac{4}{3}}(\mathcal{M}_m)S^m_4\Big]_\theta=\big[\mathrm{X}_0,\mathrm{X}_1\big]_\theta,$$

we take $(\beta_\theta,\delta_\theta)=(\beta^{1/(1-\theta)},\delta^{1/\theta})$ and define

$$f:z\in\mathcal{S}\mapsto\alpha\beta_\theta^{1-z}\gamma\delta_\theta^z\in\mathrm{X}_0+\mathrm{X}_1.$$

Since f is analytic and $f(\theta) = xd_\varphi^{\frac{q+1}{2q}}$, we conclude

$$\big(f_{|\partial_0}, f_{|\partial_1}\big) \in \big(xd_\varphi^{\frac{p+1}{2p}} \otimes 1, xd_\varphi^{\frac{3}{4}} \otimes 1\big) + \mathcal{H}'_{r,0}.$$

Therefore, taking $f_{|\partial_j} = f_j$ we obtain the following estimate

$$\begin{aligned}
&\Big\| w_c\big(xd_\varphi^{\frac{q+1}{2q}}\big)\Big\|_{S_1^m(\mathcal{H}^r_{2p,2}/\mathcal{H}'_{r,0})} \\
&\le \max\Big\{\big\|f_0\big\|_{S_1^m(L^r_{\frac{2p}{p+1}}(\mathcal{M})\otimes_h L_2^{rp}(\partial_0))}, \big\|f_1\big\|_{S_1^m(L^r_{\frac{4}{3}}(\mathcal{M})\otimes_h L_2^{oh}(\partial_1))}\Big\} \\
&= \max\Big\{\big\|\alpha\beta_\theta\gamma\big\|_{S_1^m(L^r_{\frac{2p}{p+1}}(\mathcal{M}))}, \big\|\alpha\gamma\delta_\theta\big\|_{S_1^m(L^r_{\frac{4}{3}}(\mathcal{M}))}\Big\} \le 1,
\end{aligned}$$

where the last identity follows from (4.7) and the fact that $(\alpha\beta_\theta, \gamma\delta_\theta)$ are in the unit balls of $L_{2p/p+1}(\mathcal{M}_m)$ and S_4^m respectively. The assertion for the mapping w_r is proved similarly. The proof is complete. □

Proposition 4.25. *Let $\mathcal{M}$ be a finite von Neumann algebra equipped with a n.f. state φ and let d_φ be the associated density. If $2 \le q' < p'$ and $\frac{1}{2q'} = \frac{1-\theta}{2p'} + \frac{\theta}{4}$, we have complete isomorphisms*

$$u_r : d_\varphi^{\frac{1}{2q'}} x \in L^r_{2q'}(\mathcal{M}) \mapsto \Big(1 \otimes d_\varphi^{\frac{1}{2p'}} x, 1 \otimes d_\varphi^{\frac{1}{4}} x\Big) + \mathcal{H}_{r,0} \in \mathcal{H}^r_{2p',2}(\mathcal{M},\theta)/\mathcal{H}_{r,0},$$

$$u_c : xd_\varphi^{\frac{1}{2q'}} \in L^c_{2q'}(\mathcal{M}) \mapsto \Big(xd_\varphi^{\frac{1}{2p'}} \otimes 1, xd_\varphi^{\frac{1}{4}} \otimes 1\Big) + \mathcal{H}_{c,0} \in \mathcal{H}^c_{2p',2}(\mathcal{M},\theta)/\mathcal{H}_{c,0}.$$

Proof. This follows easily from the identity

$$\mathrm{tr}_{\mathcal{M}}(x^*y) = \int_{\partial\mathcal{S}} \mathrm{tr}_{\mathcal{M}}\big(f(\overline{z})^* g(z)\big) d\mu_\theta(z),$$

valid for any pair of analytic functions f and g such that $(f(\theta), g(\theta)) = (x, y)$. Indeed, according to the definition of the mappings u_r, u_c, w_r, w_c, this means that we have

$$\begin{aligned}
\big\langle u_r(x_1), w_r(y_1)\big\rangle &= \langle x_1, y_1\rangle \\
\big\langle u_c(x_2), w_c(y_2)\big\rangle &= \langle x_2, y_2\rangle
\end{aligned}$$

for any

$$(x_1, x_2, y_1, y_2) \in L^r_{2q'}(\mathcal{M}) \times L^c_{2q'}(\mathcal{M}) \times L^c_{\frac{2q}{q+1}}(\mathcal{M}) \times L^r_{\frac{2q}{q+1}}(\mathcal{M}).$$

In particular, we deduce

$$w_r^* u_r = id_{L^r_{2q'}(\mathcal{M})} \quad \text{and} \quad w_c^* u_c = id_{L^c_{2q'}(\mathcal{M})}.$$

Therefore, the assertion follows from Lemma 4.23 and Lemma 4.24. □

Let $\mathcal{M}$ be a von Neumann algebra equipped with a *n.f.* state φ and let $\mathcal{M}_m$ be the tensor product $\mathrm{M}_m \otimes \mathcal{M}$. Then, if $\mathsf{E}_m = id_{\mathrm{M}_m} \otimes \varphi : \mathcal{M}_m \to \mathrm{M}_m$ denotes the associated conditional expectation, the following generalizes Lemma 3.6.

Lemma 4.26. *If $1/r = 1/q - 1/p$, we have isometries*

$$\begin{aligned}
L^{2p}_{(2r,\infty)}(\mathcal{M}_m, \mathsf{E}_m) &= C_p^m \otimes_h L^r_{2q}(\mathcal{M}) \otimes_h R_m, \\
L^{2p}_{(\infty,2r)}(\mathcal{M}_m, \mathsf{E}_m) &= C_m \otimes_h L^c_{2q}(\mathcal{M}) \otimes_h R_p^m.
\end{aligned}$$

Proof. By Kouba's theorem, the spaces on the right form complex interpolation families with respect to both indices p and q. Let us see that the same happens for the conditional L_p spaces on the left. Indeed, if we fix p and move $1 \le q \le p$ so that $1/q = 1 - \theta + \theta/p$, we have to justify the isometries

$$\begin{aligned} L_{(2r,\infty)}^{2p}(\mathcal{M}_m, \mathsf{E}_m) &= \Big[L_{(2p',\infty)}^{2p}(\mathcal{M}_m, \mathsf{E}_m), L_{(\infty,\infty)}^{2p}(\mathcal{M}_m, \mathsf{E}_m) \Big]_\theta, \\ L_{(\infty,2r)}^{2p}(\mathcal{M}_m, \mathsf{E}_m) &= \Big[L_{(\infty,2p')}^{2p}(\mathcal{M}_m, \mathsf{E}_m), L_{(\infty,\infty)}^{2p}(\mathcal{M}_m, \mathsf{E}_m) \Big]_\theta. \end{aligned}$$

As far as p is finite this is part of Theorem 3.2, while the remaining case follows from Lemma 2.5. This means that it suffices to prove the assertion for $q = 1$ since the case $q = p$ follows from the trivial isometries

$$\begin{aligned} L_{2p}(\mathcal{M}_m) &= L_{2p}^r(\mathcal{M}_m) = C_p^m \otimes_h L_{2p}^r(\mathcal{M}) \otimes_h R_m, \\ L_{2p}(\mathcal{M}_m) &= L_{2p}^c(\mathcal{M}_m) = C_m \otimes_h L_{2p}^c(\mathcal{M}) \otimes_h R_p^m. \end{aligned}$$

Now we claim that we also have

$$\begin{aligned} L_{(2p',\infty)}^{2p}(\mathcal{M}_m, \mathsf{E}_m) &= \Big[L_{(\infty,\infty)}^{2}(\mathcal{M}_m, \mathsf{E}_m), L_{(2,\infty)}^{\infty}(\mathcal{M}_m, \mathsf{E}_m) \Big]_{\frac{1}{p'}}, \\ L_{(\infty,2p')}^{2p}(\mathcal{M}_m, \mathsf{E}_m) &= \Big[L_{(\infty,\infty)}^{2}(\mathcal{M}_m, \mathsf{E}_m), L_{(\infty,2)}^{\infty}(\mathcal{M}_m, \mathsf{E}_m) \Big]_{\frac{1}{p'}}. \end{aligned}$$

Indeed, recalling that

$$L_{(\infty,\infty)}^{2}(\mathcal{M}_m, \mathsf{E}_m) = L_2(\mathcal{M}_m),$$

we deduce that both spaces above are reflexive. Therefore, using Theorem 3.2 (b) for amalgamated L_p spaces and duality, we deduce our claim and we are reduced to show that

$$L_{(2,\infty)}^{\infty}(\mathcal{M}_m, \mathsf{E}_m) = \mathrm{M}_m\big(L_2^r(\mathcal{M})\big) \quad \text{and} \quad L_{(\infty,2)}^{\infty}(\mathcal{M}_m, \mathsf{E}_m) = \mathrm{M}_m\big(L_2^c(\mathcal{M})\big).$$

However, these isometries are exactly (2.9). The proof is complete. □

Let $\mathcal{M}$ be a von Neumann algebra equipped with a *n.f.* state φ and let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$. Let $\mathsf{E} : \mathcal{M} \to \mathcal{N}$ denote the corresponding conditional expectation. In Section 3, we defined the spaces

$$\begin{aligned} \mathcal{R}_{2p,q}^n(\mathcal{M}, \mathsf{E}) &= n^{\frac{1}{2p}} L_{2p}^r(\mathcal{M}) \cap n^{\frac{1}{2q}} L_{(2r,\infty)}^{2p}(\mathcal{M}, \mathsf{E}), \\ \mathcal{C}_{2p,q}^n(\mathcal{M}, \mathsf{E}) &= n^{\frac{1}{2p}} L_{2p}^c(\mathcal{M}) \cap n^{\frac{1}{2q}} L_{(\infty,2r)}^{2p}(\mathcal{M}, \mathsf{E}), \end{aligned}$$

and mentioned the isomorphism from [16]

$$\mathcal{J}_{p,q}^n(\mathcal{M}, \mathsf{E}) \simeq \mathcal{R}_{2p,q}^n(\mathcal{M}, \mathsf{E}) \otimes_{\mathcal{M}} \mathcal{C}_{2p,q}^n(\mathcal{M}, \mathsf{E}). \tag{4.8}$$

In fact, to be completely fair we should say that we have slightly modified the definition of $\mathcal{R}_{2p,q}^n(\mathcal{M}, \mathsf{E})$ and $\mathcal{C}_{2p,q}^n(\mathcal{M}, \mathsf{E})$ by considering the row/column o.s.s. of $L_{2p}(\mathcal{M})$. However, the new definition coincides with the former one in the Banach space level. Hence, since we do not even have an operator space structure for these spaces, our modification is only motivated for notational convenience below. Namely, inspired by Lemma 4.26, we introduce the operator spaces

$$\begin{aligned} \mathcal{R}_{2p,q}^n(\mathcal{M}) &= n^{\frac{1}{2p}} L_{2p}^r(\mathcal{M}) \cap n^{\frac{1}{2q}} L_{2q}^r(\mathcal{M}), \\ \mathcal{C}_{2p,q}^n(\mathcal{M}) &= n^{\frac{1}{2p}} L_{2p}^c(\mathcal{M}) \cap n^{\frac{1}{2q}} L_{2q}^c(\mathcal{M}). \end{aligned}$$

These spaces give rise to the complete isomorphism

$$\mathcal{J}_{p,q}^n(\mathcal{M}) \simeq_{cb} \mathcal{R}_{2p,q}^n(\mathcal{M}) \otimes_{\mathcal{M},h} \mathcal{C}_{2p,q}^n(\mathcal{M}). \tag{4.9}$$

Indeed, taking $(\mathcal{M}, \mathcal{N}, \mathsf{E}) = (\mathcal{M}_m, \mathrm{M}_m, \mathsf{E}_m)$ in (4.8) we have

$$\begin{aligned} S_p^m\big(\mathcal{J}_{p,q}^n(\mathcal{M})\big) & = \mathcal{J}_{p,q}^n(\mathcal{M}_m, \mathsf{E}_m) \\ & \simeq \mathcal{R}_{2p,q}^n(\mathcal{M}_m, \mathsf{E}_m) \otimes_{\mathcal{M}_m} \mathcal{C}_{2p,q}^n(\mathcal{M}_m, \mathsf{E}_m) \\ & = S_{(2p,\infty)}^m\big(\mathcal{R}_{2p,q}^n(\mathcal{M})\big) \otimes_{\mathcal{M}_m} S_{(\infty,2p)}^m\big(\mathcal{C}_{2p,q}^n(\mathcal{M})\big) \\ & = C_p^m \otimes_h \Big(\mathcal{R}_{2p,q}^n(\mathcal{M}) \otimes_{\mathcal{M},h} \mathcal{C}_{2p,q}^n(\mathcal{M})\Big) \otimes_h R_p^m \\ & = S_p^m\big(\mathcal{R}_{2p,q}^n(\mathcal{M}) \otimes_{\mathcal{M},h} \mathcal{C}_{2p,q}^n(\mathcal{M})\big), \end{aligned}$$

where the third identity follows from Lemma 4.26 after taking in consideration our *new* definition of $\mathcal{R}_{2p,q}^n(\mathcal{M}, \mathsf{E})$ and $\mathcal{C}_{2p,q}^n(\mathcal{M}, \mathsf{E})$. In other words, (4.8) and (4.9) are the amalgamated and operator space versions of the same factorization isomorphism. Now assume that $\mathcal{M}$ is equipped with a *n.s.s.f.* weight ψ, given by the increasing sequence $(\psi_n, q_n)_{n\geq 1}$. Then, we may generalize the factorization result above in the usual way. Namely, assuming by approximation that $\mathrm{k}_n = \psi_n(1)$ are positive integers, we define

$$\begin{aligned} \mathcal{R}_{2p,q}(\psi_n) & = \mathrm{k}_n^{\frac{1}{2p}} L_{2p}^r(q_n \mathcal{M} q_n) \cap \mathrm{k}_n^{\frac{1}{2q}} L_{2q}^r(q_n \mathcal{M} q_n), \\ \mathcal{C}_{2p,q}(\psi_n) & = \mathrm{k}_n^{\frac{1}{2p}} L_{2p}^c(q_n \mathcal{M} q_n) \cap \mathrm{k}_n^{\frac{1}{2q}} L_{2q}^c(q_n \mathcal{M} q_n). \end{aligned}$$

This gives the complete isomorphism

$$\mathcal{J}_{p,q}(\psi_n) \simeq_{cb} \mathcal{R}_{2p,q}(\psi_n) \otimes_{\mathcal{M},h} \mathcal{C}_{2p,q}(\psi_n) = \bigcap_{u,v \in \{2p,2q\}} \mathrm{k}_n^{\frac{1}{u}+\frac{1}{v}} L_{(u,v)}(q_n \mathcal{M} q_n).$$

Then, taking direct limits we obtain the space

$$\mathcal{J}_{p,q}(\psi) = \mathcal{R}_{2p,q}(\psi) \otimes_{\mathcal{M},h} \mathcal{C}_{2p,q}(\psi) = \bigcap_{u,v \in \{2p,2q\}} L_{(u,v)}(\mathcal{M}).$$

Lemma 4.27. *Let $\mathcal{M}$ be a von Neumann algebra equipped with a n.s.s.f. weight ψ. Then, there exists a n.s.s.f. weight ξ on $\mathcal{B}(\ell_2)$ such that the following complete isomorphisms hold*

$$\begin{aligned} \mathcal{H}_{2p',2}^r(\mathcal{M}, \theta) \otimes_h R & \simeq_{cb} \mathcal{R}_{2p',2}(\psi \otimes \xi), \\ C \otimes_h \mathcal{H}_{2p',2}^c(\mathcal{M}, \theta) & \simeq_{cb} \mathcal{C}_{2p',2}\,(\psi \otimes \xi). \end{aligned}$$

Proof. By symmetry, we only consider the column case. Let us first observe that $\mathcal{H}_2$ is indeed the graph of an injective closed densely-defined (unbounded) operator with dense range. This is quite similar to Remark 1.3. It follows from the three lines lemma that for $z = a + ib$

$$|f(z)| \leq \|f_{|\partial_0}\|_{L_2(\partial_0, \mu_a)}^{1-a} \|f_{|\partial_1}\|_{L_2(\partial_1, \mu_a)}^{a}.$$

Since μ_a and μ_θ have the same null sets, we deduce that

$$\pi_j(f) = f_{|\partial_j} \in L_2(\partial_j, \mu_\theta)$$

are injective for $j = 0, 1$ when restricted to analytic functions. Thus, the mapping $\Lambda(\pi_0(f)) = \pi_1(f)$ is an injective closed densely-defined operator with dense range

and $\mathcal{H}_2$ is its graph. Let $\Lambda = u|\Lambda|$ be the polar decomposition. Since $M_u = 1 \otimes u^*$ defines a complete isometry (recall that Λ has dense range)

$$L_4^c(\mathcal{M}) \otimes_h L_2^{oh}(\partial_1) \to L_4^c(\mathcal{M}) \otimes_h L_2^{oh}(\partial_0),$$

we may replace Λ by $|\Lambda|$ in the definition of $\mathcal{H}_{2p',2}^c(\mathcal{M},\theta)$. Using the discretization Lemma 4.5, we may also replace $L_2(\partial_0)$ by ℓ_2 and the operator $|\Lambda|$ by a diagonal operator d_λ. These considerations provide a cb-isomorphism

$$C \otimes_h \mathcal{H}_{2p',2}^c(\mathcal{M},\theta) \simeq_{cb} \Big(C \otimes_h L_{2p'}^c(\mathcal{M}) \otimes_h R_{p'} \Big) \cap \Big(C \otimes_h L_4^c(\mathcal{M}) \otimes_h \ell_2^{oh}(\lambda) \Big),$$

where $\ell_2^{oh}(\lambda)$ is the weighted form of OH which arises from the action of d_λ. The assertion follows by a direct limit argument. Indeed, the *n.s.s.f.* weight ψ on $\mathcal{M}$ is given by the sequence $(\psi_n, q_n)_{n\ge1}$. On the other hand, we may consider the *n.s.s.f.* weight ξ on $\mathcal{B}(\ell_2)$ determined by the sequence $(\xi_n, \pi_n)_{n\ge1}$, where π_n is the projection onto the first n coordinates and ξ_n is the finite weight on $\pi_n \mathcal{B}(\ell_2)\pi_n$ given by

$$\xi_n \Big(\pi_n \big(\sum\nolimits_{ij} x_{ij} e_{ij} \big) \pi_n \Big) = \sum_{k=1}^n \gamma_k x_{kk} \quad \text{with} \quad \gamma_k^{\frac14 - \frac{1}{2p'}} = \lambda_k.$$

Let us define the parameters $\mathrm{k}_n' = \xi_n(1)$ and $\mathrm{w}_n = \mathrm{k}_n \mathrm{k}_n'$. Then, arguing as we did in Lemma 4.7, it turns out that the intersection space above is the direct limit of the following sequence of spaces

$$\mathrm{w}_n^{\frac{1}{2p'}} \Big(L_{2p'}^c \big(q_n \mathcal{M} q_n \bar{\otimes} \pi_n \mathcal{B}(\ell_2) \pi_n \big) \Big) \cap \mathrm{w}_n^{\frac14} \Big(L_4^c \big(q_n \mathcal{M} q_n \bar{\otimes} \pi_n \mathcal{B}(\ell_2) \pi_n \big) \Big).$$

However, the latter space is $\mathcal{C}_{2p',2}(\psi_n \otimes \xi_n)$. This completes the proof. □

Proposition 4.28. *The predual space of*

$$\mathcal{H}_{2p',2}^r(\mathcal{M},\theta) \otimes_{\mathcal{M},h} \mathcal{H}_{2p',2}^c(\mathcal{M},\theta)$$

embeds completely isomorphically into $\mathcal{K}_{rc_p}^p(\phi \otimes \psi \otimes \xi)$ *for some n.s.s.f. weight* ξ *on* $\mathcal{B}(\ell_2)$ *and where* ϕ *denotes the quasi-free state over the hyperfinite* III_1 *factor* $\mathcal{R}$ *considered in Proposition 4.16.*

Proof. According to Lemma 4.27

$$\begin{aligned} & \mathcal{H}_{2p',2}^r(\mathcal{M},\theta) \otimes_{\mathcal{M},h} \mathcal{H}_{2p',2}^c(\mathcal{M},\theta) \\ = \quad & \Big(\mathcal{H}_{2p',2}^r(\mathcal{M},\theta) \otimes_h R \Big) \otimes_{\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2),h} \Big(C \otimes_h \mathcal{H}_{2p',2}^c(\mathcal{M},\theta) \Big) \\ \simeq_{cb} \quad & \mathcal{R}_{2p',2}(\psi \otimes \xi) \otimes_{\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2),h} \mathcal{C}_{2p',2}(\psi \otimes \xi) \ \simeq_{cb} \ \mathcal{J}_{p',2}(\psi \otimes \xi). \end{aligned}$$

However, $\mathcal{J}_{p',2}(\psi \otimes \xi)$ is a direct limit of spaces

$$\mathcal{J}_{p',2}(\psi_n \otimes \xi_n) = \mathcal{J}_{p',2}^{\mathrm{w}_n} \Big(q_n \mathcal{M} q_n \bar{\otimes} \pi_n \mathcal{B}(\ell_2) \pi_n \Big).$$

According to Proposition 4.16, the direct limit

$$\mathcal{K}_{p,2}(\psi \otimes \xi) = \lim\nolimits_n \mathcal{K}_{p,2}^n(\psi_n \otimes \xi_n)$$

of the corresponding predual spaces cb-embeds into $\mathcal{K}_{rc_p}^p(\phi \otimes \psi \otimes \xi)$. □

Now we are ready to prove our main result. In the proof we shall need to work with certain quotient of $\mathcal{H}_{2p',2}^r(\mathcal{M},\theta) \otimes_{\mathcal{M},h} \mathcal{H}_{2p',2}^c(\mathcal{M},\theta)$. Namely, recalling the subspaces $\mathcal{H}_{r,0}$ and $\mathcal{H}_{c,0}$, we set

$$\mathcal{Q}_{2p',2}(\mathcal{M},\theta) = \big(\mathcal{H}_{2p',2}^r(\mathcal{M},\theta) / \mathcal{H}_{r,0} \big) \otimes_{\mathcal{M},h} \big(\mathcal{H}_{2p',2}^c(\mathcal{M},\theta) / \mathcal{H}_{c,0} \big).$$

We claim that $\mathcal{Q}_{2p',2}(\mathcal{M},\theta)$ is a quotient of $\mathcal{H}^r_{2p',2}(\mathcal{M},\theta)\otimes_{\mathcal{M},h}\mathcal{H}^c_{2p',2}(\mathcal{M},\theta)$. Indeed, according to the definition of the $\mathcal{M}$-amalgamated Haagerup tensor product of two operator spaces (see Remark 3.8), we may write $\mathcal{Q}_{2p',2}(\mathcal{M},\theta)$ as a quotient of the Haagerup tensor product

$$\Lambda_{2p',2}(\mathcal{M},\theta) = \big(\mathcal{H}^r_{2p',2}(\mathcal{M},\theta)/\mathcal{H}_{r,0}\big) \otimes_h \big(\mathcal{H}^c_{2p',2}(\mathcal{M},\theta)/\mathcal{H}_{c,0}\big)$$

by the closed subspace spanned by the differences $x_1\gamma \otimes x_2 - x_1 \otimes \gamma x_2$, with $\gamma \in \mathcal{M}$. Therefore, it suffices to see that the space $\Lambda_{2p',2}(\mathcal{M},\theta)$ is a quotient of $\mathcal{H}^r_{2p',2}(\mathcal{M},\theta) \otimes_h \mathcal{H}^c_{2p',2}(\mathcal{M},\theta)$. However, this follows from the projectivity of the Haagerup tensor product and our claim follows.

Theorem 4.29. *Let $1 \le p < q \le 2$ and let $\mathcal{M}$ be a von Neumann algebra. Then, there exists a sufficiently large von Neumann algebra $\mathcal{A}$ and a completely isomorphic embedding of $L_q(\mathcal{M})$ into $L_p(\mathcal{A})$, where both spaces are equipped with their respective natural operator space structures. Moreover, we have*

- **(a)** *If $\mathcal{M}$ is* QWEP*, we can choose $\mathcal{A}$ to be* QWEP.
- **(b)** *If $\mathcal{M}$ is hyperfinite, we can choose $\mathcal{A}$ to be hyperfinite.*

Proof. Let us first assume that $\mathcal{M}$ is finite. According to Theorem 4.17 and Proposition 4.28, it suffices to prove that $L_{q'}(\mathcal{M})$ is completely isomorphic to a quotient of $\mathcal{H}^r_{2p',2}(\mathcal{M},\theta) \otimes_{\mathcal{M}} \mathcal{H}^c_{2p',2}(\mathcal{M},\theta)$. This follows from Proposition 4.25 since

$$L_{q'}(\mathcal{M}) \simeq_{cb} L^r_{2q'}(\mathcal{M}) \otimes_{\mathcal{M},h} L^c_{2q'}(\mathcal{M}) \simeq_{cb} \mathcal{Q}_{2p',2}(\mathcal{M},\theta).$$

The construction of the cb-embedding for a general von Neumann algebra $\mathcal{M}$ can be obtained by using Haagerup's approximation theorem [5] and the fact that direct limits are stable in our construction. Indeed, Haagerup theorem shows that for every σ-finite von Neumann algebra $\mathcal{M}$, the space $L_q(\mathcal{M})$ is complemented in a direct limit of L_q spaces over finite von Neumann algebras. Finally, if $\mathcal{M}$ is any von Neumann algebra, we observe that $L_q(\mathcal{M})$ can always be written as a direct limit of L_q spaces associated to σ-finite von Neumann algebras. On the other hand, the stability of hyperfiniteness follows directly from our construction. Indeed, our construction goes as follows

$$L_q(\mathcal{M}) \to \Big(\mathcal{H}^r_{2p',2}(\mathcal{M},\theta) \otimes_{\mathcal{M},h} \mathcal{H}^c_{2p',2}(\mathcal{M},\theta)\Big)_* \to \mathcal{K}^p_{rc_p}(\phi \otimes \psi \otimes \xi) \to L_p(\mathcal{A})$$

where the first embedding follows as above, the second from Proposition 4.28 and the last one from Theorem 4.17. In particular, it turns out that the von Neumann algebra $\mathcal{A}$ is of the form

$$\mathcal{A} = M_s\big(\mathcal{R}\bar{\otimes}\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2)\big),$$

which is hyperfinite when $\mathcal{M}$ is hyperfinite and is a factor when $\mathcal{M}$ is a factor. Finally, it remains to justify that the QWEP is preserved. If $\mathcal{M}$ is QWEP, there exists a completely isometric embedding of $L_q(\mathcal{M})$ into $L_q(\mathcal{M}_\mathcal{U})$ with $\mathcal{M}_\mathcal{U}$ of the form

$$\mathcal{M}_\mathcal{U} = \Big(\prod\nolimits_{n,\mathcal{U}} S_1^n\Big)^*.$$

Since we know from Corollary 4.8 that the Schatten class S_q^n embeds completely isomorphically into $L_p(\mathcal{A}_n)$ with relevant constants independent of n and $\mathcal{A}_n$ being QWEP, we find a completely isomorphic embedding

$$L_q(\mathcal{M}) \to L_p(\mathcal{A}_\mathcal{U}) \quad \text{with} \quad \mathcal{A}_\mathcal{U} = \Big(\prod\nolimits_{n,\mathcal{U}} \mathcal{A}_{n*}\Big)^*.$$

This proves the assertion since $\mathcal{A}_\mathcal{U}$ is QWEP. The proof is complete. $\square$

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